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CURRENT MISUSE OF THE FACTORIAL METHODS

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Papers involving factorial methods appearing in current literature often misuse factorial methods. These meaningless results are reached because of neglect of certain conditions basic to factor theory. The conditions are:

(1) *The number of basic factors must be smaller than the number of tests.*

(2) *The diagonals of the correlation matrix must be regarded as unknown.*

(3) *The axes must be rotated into a simple configuration.*

(4) *Each factor must be overdetermined by appearance in several tests.*

(5) *Tests should have simple factorial composition.*

Those psychologists who have devoted themselves recently to the development of factor analysis should be encouraged by the very general interest in these new methods. The number of papers in the psychological journals that involve factor analysis is increasing rapidly. It is therefore a serious matter that the large majority of these papers involve misinterpretation of the factorial methods. Factor theory is still imperfect and there are challenging theoretical problems for the mathematicians to solve in making these new analytical tools even more powerful than they now are. But if the misapplication of factor methods continue at the present rate, we shall soon find general disappointment with the results because they are usually meaningless as far as psychological interpretation is concerned.

With the hope of assisting in the best use of these new methods in their present state of development I shall summarize what seems to me to be the most common reasons for the meaningless results that are reported in many of the current factorial studies.

1. The number of psychologically basic factors in a battery of tests must be considerably smaller than the number of tests in the battery.

The reasonableness of this restriction can be seen without recourse to mathematical proofs. If we must postulate a new factor for every test that we add to a battery, it becomes impossible to identify the factors with confidence. This restriction has been dealt with for-

mally but it reduces to a universal principle in science, namely that a hypothesis must be overdetermined by the experimental data before it is demonstrated as plausible. This principle is so universal that we use it intuitively in many situations.

A very simple application of this principle is the fitting of a curve to some experimental data. If there are three parameters in the curve we certainly want more than three observations. If every test in the battery introduces a new common factor we have a system that is too complex for a unique determination of the common factors.

I do not know of any sure way to guarantee that a given test battery will satisfy this condition. We must rely on psychological intuition to assemble a test battery with sufficient restriction in content so as not to cover too many common factors. If we have too complex a test battery then we must either add more tests to the battery, or else reduce the number by retaining a more or less homogeneous group of tests for factorial analysis. The more scattered the content of the tests, the longer must be the test battery, and the longer will be the computational work in extracting a large number of factors.

2. The diagonals must be regarded as unknown communalities unless one can be certain that specific factors are absent.

This is a question about which the students of factor analysis are not all agreed. Since this question about the diagonal entries is still debated, I shall present this restriction as my own interpretation of the problem. There are certain conditions in which a psychologically meaningful solution can be found even when the communalities are not used in the diagonal cells of the correlational matrix. If the correlations are corrected for attenuation, and if the test battery has been so assembled that specific factors can be ignored, then it is possible to factor a correlational matrix with unity in the diagonals and with side entries that have been corrected for attenuation. Occasionally this procedure can be justified. If we have a large number of variables such as thirty or forty or more, and if we extract only two or three factors, it is possible that the distortion from unity in the diagonals will be small enough so that it can be ignored. The simplest procedure is to estimate the communalities for each successive factor.

3. No matter how the correlational matrix is factored, the axes must be rotated into a simple configuration before any psychological interpretation can be made. The frequent attempt to find psychological interpretation for the centroid axes without rotation, and for the principal components without rotation, are examples of this error.

This is the most serious and the most frequent error in current factorial studies. Time after time the author of an article reports that his psychological tests show a single common factor with saturations that are all positive. The subsequent factors all have some positive and some negative saturations, and the interpretation of these subsequent factors is left indeterminate. Sometimes an attempt is made to put psychological meaning into the second and subsequent factors but always with some hazy reservations about the fact that the saturations are both positive and negative.

The problem can be regarded either algebraically or geometrically according to one's preferences, but the essential argument is the same. I have used geometrical interpretations because they have seemed to be the clearest.

The factorial description of a test must remain invariant when the test is moved from one battery to another. This simply means that if a test calls for a certain amount of one ability and a certain amount of another ability, then that test should be so described irrespective of the other tests in the battery. If we ask the subjects to take some more tests at a later time, and if we add these tests to the battery, then the principal components and the centroid axes will be altered, and the factorial description of each test will also be altered. In order that the factorial description shall make psychological sense, the factors that enter into a performance must be independent of the other tests that we might ask the subjects to take. The minor deviations from this principle that can be introduced by contrast effects, or by suggestion, or by practice, are trivial in comparison with the gross effects that are determined by shifting from the principal axes of one test battery to those of another test battery. If the test battery is described in terms of the primary axes of a simple configuration, then the factorial description of each test remains invariant when some tests are removed and other tests added. This is a psychological requirement that is independent of any mathematical or statistical arguments.

4. No meaningful component can be identified unless each factor is overdetermined with three or four or more tests. In order to make a psychological identification of a common factor it is essential to have a number of different tests that involve the factor. Only in this manner can we have confidence in identifying the nature of each common factor. Merely to know that certain tests have one or more factors in common is not satisfactory for psychological interpretation. We want to know the psychological nature of each factor.

5. In order that meaningful factors shall emerge, it is essential that the individual tests be as simple as possible as regards factorial composition. If all of the tests involve many psychological factors, then the identification of the factors becomes very difficult. Composite tests like the Stanford Binet and composite group tests of intelligence are so complex in factorial composition that they are not useful in the identification of basic factors unless they are analyzed by separate items.

While the factorial methods are not yet perfected, they are sufficiently developed so that they can be used in the solution of many fundamentally important psychological problems. The factorial methods must be so formulated that they satisfy not only the requirements of mathematical rigor but also the restrictions of the psychological problems which they are intended to solve. Some of the conditions here listed are not generally accepted, but the writer is quite certain that psycholologically meaningful results will not be obtained unless we satisfy these conditions. Final judgment about the conditions here listed will be in terms of psychological results. The factorial methods must justify themselves in testing psychological hypotheses and in producing psychologically meaningful and significant discoveries.

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A TABLE TO AID IN THE COMPUTATION OF FISHER'S "t" FUNCTION FOR COMPARISON OF TWO MEANS

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A table to facilitate the computation of Fisher's t function for the comparison of two means is presented. The derivations leading to the construction of the table, and the necessary instructions for its use, are given.

After using Fisher's "t" function* for comparison of two means, it seemed that a rearrangement of the formula and the construction of a facilitating table would make such computation much more rapid. The formula may be rewritten:

$$t = \frac{M_1 - M_2}{\sqrt{N_1\sigma_1^2 + N_2\sigma_2^2}} \sqrt{N_1N_2 - \frac{2N_1N_2}{N_1 + N_2}}$$

where M_1 and M_2 are the means of the samples, N_1 and N_2 the numbers of observations, and σ_1^2 and σ_2^2 the respective variances. The values of the radical function of N_1 and N_2, for values of N_1 and N_2 from 5 to 50, are to be found in the accompanying table.

The use of the rearranged function and the table is as follows.

1. Obtain ΣX and ΣX^2 for each sample.

2. Calculate the respective means and their difference.

3. The expression $N\sigma^2$ for each sample may be easily computed from the relationship $N\sigma^2 = [N\Sigma X^2 - (\Sigma X)^2]/N$. The square root of the sum $(N_1\sigma_1^2 + N_2\sigma_2^2)$ should be obtained.

4. Divide the difference of the means by the root just obtained.

5. Look up the value of the radical function of N_1 and N_2 in the table. Let the larger N determine the column of the table and the smaller N determine the row. The value of "t" is the product of the table entry and the quotient obtained in step 4. Interpretation of the obtained value of "t" may be made by referring to Fisher's table.

*Fisher, R. A., *Statistical Methods for Research Workers.* Sixth Edition, Edinburgh, 1936, page 128.

Construction of the table:

The table was constructed by first entering on the computation sheets the various values of $N_1 + N_2$. Immediately below each was placed the corresponding value of (N_1N_2). The quotionts $(2N_1N_2)/(N_1 + N_2)$ were computed to eleven decimals, so that the square roots would be accurate to five decimals. The quotients were subtracted from N_1N_2 and the square root of the differences obtained. All work was checked as follows: $(N_1 + N_2)$ by readdition, the products N_1N_2 by remultiplying, using N_2 as the multiplier instead of the multiplicand as was done in the computation. The quotient and difference were checked by noting that the function may also be written,

$$\sqrt{\frac{N_1N_2(N_1+N_2-2)}{N_1+N_2}}$$

The product indicated in the numerator was obtained and immediately divided by $(N_1 + N_2)$ before clearing the calculator. The square roots were checked by squaring.

	5	6	7	8	9	10	11	12	13
5	4.47214	4.95434	5.40062	5.81774	6.21059	6.58281	6.93722	7.27607	7.60117
6		5.47723	5.96141	6.41427	6.84105	7.24569	7.63121	8.00000	8.35401
7			6.48074	6.96659	7.42462	7.85905	8.27312	8.66937	9.04986
8				7.48331	7.97053	8.43274	8.87338	9.29516	9.70027
9					8.48528	8.97365	9.43928	9.88505	10.31327
10						9.48683	9.97616	10.44466	10.89475
11							10.48809	10.97824	11.44916
12								11.48913	11.97998
13									12.49000
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5	7.91401	8.21584	8.50770	8.79049	9.06498	9.33185	9.59166	9.84495
6	8.69483	9.02378	9.34199	9.65041	9.94987	10.24109	10.52470	10.80123
7	9.41630	9.77008	10.11241	10.44430	10.76661	11.08013	11.38550	11.68332
8	10.09050	10.46734	10.83205	11.18571	11.52923	11.86342	12.18899	12.50655
9	10.72583	11.12430	11.51000	11.88406	12.24745	12.60102	12.94551	13.28157
10	11.32843	11.74734	12.15287	12.54621	12.92837	13.30025	13.66260	14.01612
11	11.90294	12.34130	12.76569	13.17736	13.57736	13.96663	14.34596	14.71606
12	12.45299	12.90994	13.35237	13.78155	14.19859	14.60446	15.00000	15.38594
13	12.98147	13.45628	13.91600	14.36199	14.79538	15.21718	15.62826	16.02938
14	13.49074	13.98275	14.45914	14.92130	15.37043	15.80755	16.23359	16.64932
15		14.49138	14.98386	15.46165	15.92597	16.37789	16.81836	17.24819
16			15.49193	15.98484	16.46386	16.93011	17.38454	17.82801
17				16.49242	16.98571	17.46584	17.93381	18.39050
18					17.49286	17.98648	18.46761	18.93714
19						18.49324	18.98718	19.46921
20							19.49359	19.98780
21								20.49390
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5	10.09217	10.33372	10.56996	10.80123	11.02782	11.25000	11.46801	11.68206
6	11.07120	11.33502	11.59310	11.84578	12.09339	12.33620	12.57448	12.80848
7	11.97411	12.25833	12.53640	12.80869	13.07554	13.33725	13.59412	13.84638
8	12.81666	13.11980	13.41641	13.70689	13.99159	14.27085	14.54495	14.81416
9	13.60977	13.93063	14.24462	14.55214	14.85357	15.14926	15.43951	15.72460
10	14.36141	14.69900	15.02938	15.35299	15.67021	15.98141	16.28690	16.58699
11	15.07758	15.43106	15.77702	16.11590	16.44812	16.77404	17.09401	17.40833
12	15.76295	16.13160	16.49242	16.84588	17.19241	17.53239	17.86617	18.14908
13	16.42124	16.80443	17.17950	17.54693	17.90717	18.26061	18.60763	18.94855
14	17.05547	17.45264	17.84141	18.22227	18.59570	18.96209	19.32184	19.67527
15	17.66811	18.07877	18.48076	18.87459	19.26073	19.63961	20.01162	20.37713
16	18.26126	18.68497	19.09974	19.50610	19.90453	20.29549	20.67937	21.05654
17	18.83668	19.27304	19.70019	20.11870	20.52905	20.93171	21.32708	21.71555
18	19.39588	19.84452	20.28370	20.71400	21.13592	21.54994	21.95648	22.35592
19	19.94015	20.40075	20.84164	21.29340	21.72658	22.15165	22.56904	22.97916
20	20.47865	20.94289	21.40518	21.85813	22.30227	22.73810	23.16607	23.58658
21	20.98837	21.47197	21.94539	22.40924	22.86408	23.31041	23.74868	24.17933
22	21.49419	21.98889	22.47317	22.94767	23.41296	23.86954	24.31789	24.75844
23		22.49444	22.98936	23.47428	23.94978	24.41639	24.87459	25.32481
24			23.49468	23.98979	24.47529	24.95172	25.41956	25.87925
25				24.49490	24.99019	25.47623	25.95351	26.42249
26					25.49510	25.99056	26.47710	26.95518
27						26.49528	26.99091	27.47791
28							27.49545	27.99123
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5	11.89237	12.09913	12.30249	12.50263	12.69969	12.89380	13.08509	13.27368
6	13.03840	13.26446	13.48684	13.70570	13.92121	14.13351	14.34274	14.54903
7	14.09428	14.33802	14.57782	14.81384	15.04627	15.27525	15.50094	15.72346
8	15.07874	15.33891	15.59487	15.84683	16.09496	16.33942	16.58038	16.81798
9	16.00481	16.28036	16.55147	16.81836	17.08120	17.34018	17.59545	17.84718
10	16.88194	17.17201	17.45743	17.73841	18.01515	18.28782	18.55661	18.82167
11	17.71729	18.02115	18.32015	18.61451	18.90444	19.19012	19.47175	19.74947
12	18.51640	18.83342	19.14583	19.45251	19.75502	20.05312	20.34699	20.63680
13	19.28368	19.61331	19.93768	20.25704	20.57162	20.88161	21.18721	21.48860
14	20.02271	20.36446	20.70077	21.03189	21.35806	21.67948	21.99636	22.30888
15	20.73644	21.08987	21.43769	21.78015	22.11750	22.44994	22.77770	23.10095
16	21.42733	21.79205	22.15100	22.50442	22.85257	23.19567	23.53394	23.86756
17	22.09746	22.47313	22.84285	23.20690	23.56551	23.91893	24.26738	24.61105
18	22.74863	23.13492	23.51510	23.88945	24.25823	24.62167	24.97999	25.33341
19	23.38236	23.77898	24.16934	24.55371	24.93236	25.30554	25.67347	26.03638
20	24.00000	24.40669	24.80695	25.20108	25.58935	25.97201	26.34930	26.72144
21	24.60273	25.01922	25.42915	25.83280	26.23045	26.62236	27.00877	27.38991
22	25.19157	25.61765	26.03701	26.44995	26.85676	27.25770	27.65302	28.04295
23	25.76747	26.20291	26.63149	27.05352	27.46928	27.87905	28.28307	28.68159
24	26.33122	26.77584	27.21344	27.64436	28.06888	28.48729	28.89983	29.30674
25	26.88359	27.33718	27.78362	28.22325	28.65635	29.08321	29.50410	29.91925
26	27.42522	27.88762	28.34273	28.79089	29.23240	29.66756	30.09662	30.51983
27	27.95673	28.42777	28.89138	29.34791	29.79268	30.24097	30.67805	31.10918
28	28.47867	28.95818	29.43014	29.89490	30.35276	30.80404	31.24900	31.68790
29	28.99152	29.47937	29.95954	30.43237	30.89819	31.35732	31.81001	32.25655
30	29.49576	29.99180	30.48003	30.96080	31.43446	31.90129	32.36159	32.81563
31		30.49590	30.99206	31.48065	31.96200	32.43642	32.90420	33.36562
32			31.49603	31.99231	32.48123	32.96312	33.43827	33.90695
33				32.49615	32.99254	33.48178	33.96418	34.44001
34					33.49627	33.99275	34.48229	34.96517
35						34.49638	34.99296	35.48278
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5	13.45967	13.64318	13.82429	14.00311	14.17970	14.35415	14.52654	14.69694
6	14.75250	14.94326	15.15141	15.34705	15.54027	15.73116	15.91980	16.10626
7	15.94295	16.15953	16.37330	16.58438	16.79286	16.99882	17.20237	17.40358
8	17.05235	17.28362	17.51190	17.73731	17.95996	18.17993	18.39732	18.61223
9	18.09549	18.34053	18.58241	18.82126	19.05719	19.29029	19.52067	19.74842
10	19.08315	19.34119	19.59592	19.84746	20.09592	20.34143	20.58406	20.82394
11	20.02346	20.29384	20.56077	20.82436	21.08473	21.34201	21.59630	21.84769
12	20.92271	21.20488	21.48345	21.75854	22.03028	22.29880	22.56419	22.82658
13	21.78595	22.07940	22.36912	22.65523	22.93786	23.21714	23.49319	23.76610
14	22.61722	22.92152	23.22196	23.51866	23.81176	24.10139	24.38767	24.67071
15	23.41988	23.73464	24.04541	24.35232	24.65552	24.95513	25.25128	25.54408
16	24.19672	24.52160	24.84236	25.15914	25.47209	25.78135	26.08703	26.38927
17	24.95013	25.28481	25.61524	25.94158	26.26398	26.58258	26.89750	27.20887
18	25.68212	26.02630	26.36612	26.70174	27.03331	27.36097	27.68486	28.00510
19	26.39444	26.74787	27.09681	27.44145	27.78194	28.11841	28.45102	28.77988
20	27.08862	27.45103	27.80887	28.16229	28.51146	28.85651	29.19760	29.53486
21	27.76597	28.13716	28.50367	28.86565	29.22328	29.61049	29.92606	30.27150
22	28.42769	28.80744	29.18241	29.55275	29.91864	30.28023	30.63767	30.99109
23	29.07480	29.46293	29.84617	30.22468	30.59864	30.96821	31.33354	31.69478
24	29.70826	30.10458	30.49590	30.38241	31.26427	31.64165	32.01470	32.38357
25	30.32889	30.73323	31.13247	31.52680	31.91640	32.30143	32.68204	33.05839
26	30.93744	31.34964	31.75665	32.15866	32.55583	32.94836	33.33638	33.72006
27	31.53460	31.95451	32.36914	32.77867	33.18329	33.58316	33.97845	34.36932
28	32.12098	32.54847	32.97058	33.38749	33.79941	34.20650	34.60893	35.00685
29	32.69716	33.13209	33.56154	33.98571	34.40480	34.81898	35.22841	35.63327
30	33.26365	33.70589	34.14256	34.57386	35.00000	35.42115	35.83747	36.24914
31	33.82093	34.27035	34.71412	35.15245	35.58551	36.01351	36.43661	36.85498
32	34.36942	34.82593	35.27668	35.72191	36.16180	36.59654	37.02631	37.45127
33	34.90954	35.37301	35.83065	36.28267	36.72928	37.17066	37.60699	38.03844
34	35.44166	35.91199	36.37641	36.83513	37.28835	37.73627	38.17907	38.61691
35	35.96612	36.44322	36.91432	37.37963	37.83937	38.29374	38.74290	39.18705
36	36.48324	36.96701	37.44470	37.91652	38.38269	38.84341	39.29887	39.74921
37	36.99333	37.48368	37.96786	38.44610	38.91861	39.38559	39.84724	40.30372
38	37.49667	37.99351	38.48410	38.96867	39.44743	39.92060	40.38836	40.85089
39		38.49675	38.99367	39.48449	39.96943	40.44870	40.92250	41.39099
40			39.49684	39.99383	40.48487	40.97016	41.44991	41.92430
41				40.49691	40.99398	41.48522	41.97086	42.45107
42					41.49699	41.99412	42.48557	42.97152
43						42.49706	42.99425	43.48589
44							43.49713	43.99438
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5	14.86541	15.03202	15.19682	15.35988	15.52125
6	16.29063	16.47296	16.65333	16.83179	17.00840
7	17.60253	17.79929	17.99394	18.18653	18.37714
8	18.82473	19.03490	19.24281	19.44854	19.65215
9	19.97362	20.19636	20.41671	20.63475	20.85056
10	21.06114	21.29575	21.52785	21.75752	21.98484
11	22.09628	22.34217	22.58543	22.82615	23.06441
12	23.08605	23.34269	23.59661	23.84788	24.09658
13	24.03599	24.30295	24.56707	24.82844	25.08715
14	24.95062	25.22749	25.50142	25.77251	26.04083
15	25.83364	26.12007	26.40316	26.68392	26.96151
16	26.68816	26.98383	27.27636	27.56587	27.85243
17	27.51681	27.82142	28.12281	28.42108	28.71632
18	28.32181	28.63510	28.94509	29.25187	29.55553
19	29.10511	29.42685	29.74519	30.06023	30.37209
20	29.86840	30.19835	30.52482	30.84792	31.16775
21	30.61314	30.95110	31.28550	31.61645	31.91405
22	31.34063	31.68642	32.02856	32.36717	32.70236
23	32.05204	32.40547	32.75517	33.10128	33.44388
24	32.74839	33.10929	33.46640	33.81983	34.16969
25	33.43061	33.79883	34.16316	34.52379	34.88075
26	34.09953	34.47493	34.84638	35.21401	35.57794
27	34.75590	35.13834	35.51676	35.89128	36.26203
28	35.40041	35.78976	36.17501	36.55631	36.93376
29	36.03369	36.42982	36.82179	37.20973	37.59376
30	36.65630	37.05910	37.45767	37.85214	38.24265
31	37.26877	37.67812	38.08319	38.48409	38.88095
32	37.87158	38.28739	38.69884	39.10606	39.50918
33	38.46517	38.88734	39.30507	39.71852	40.12781
34	39.04997	39.47839	39.90232	40.32190	40.73725
35	39.62634	40.06093	40.49096	40.91658	41.33792
36	40.19465	40.63532	41.07137	41.50294	41.93017
37	40.75522	41.20188	41.64387	42.08132	42.51437
38	41.30836	41.76094	42.20878	42.65203	43.09081
39	41.85437	42.31279	42.76641	43.21537	43.65982
40	42.39351	42.85769	43.31701	43.77162	44.22166
41	42.92603	43.39590	43.86086	44.32105	44.77661
42	43.45217	43.92767	44.39820	44.86390	45.32491
43	43.97216	44.45322	44.92925	45.40039	45.86680
44	44.48620	44.97276	45.45423	45.93076	46.40249
45	44.99451	45.48650	45.97334	46.45519	46.93220
46	45.49725	45.99462	46.48679	46.97390	47.45612
47		46.49731	46.99474	47.48706	47.97443
48			47.49737	47.99485	48.48732
49				48.49742	48.99495
50					49.49748

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ON THE RANK OF THE REDUCED CORRELATIONAL MATRIX IN MULTIPLE-FACTOR ANALYSIS

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Since the factor problem is reduced mathematically to the expression of the obtained correlational matrix in terms of a matrix of lower rank, criteria for the determination of this lower rank are of first importance. These criteria are investigated by means of certain mathematical deductions, and brought into relation with Spearman's and Thurstone's factor theories.

I

One of the principal problems in the mathematical theory of Multiple-Factor Analysis is the discussion of the reduced correlational matrix*

$$R = \begin{vmatrix} x_1 & r_{12} \cdots\cdots r_{1n} \\ r_{21} & x_2 \quad \cdots\cdots r_{2n} \\ & \\ r_{n1} & r_{n2} \cdots\cdots x_n \end{vmatrix}, \tag{1.1}$$

a symmetrical matrix ($r_{ik} = r_{ki}$) of order n, when n tests are to be analyzed. The coefficients $r_{ik} (i \neq k)$ are the observed intercorrelations of the tests, while the diagonal elements $r_{ii} = x_i$, or the *communalities,* are, at the outset, unknown quantities which must, however, lie between zero and unity. A set of communalities is then chosen such that the rank of R becomes as low as possible. This minimum rank will be denoted by ϱ. The question now arises: How far can the rank be reduced by a suitable choice of the diagonal elements x_i when the non-diagonal elements $r_{ik} (i \neq k)$ are given, but *arbitrary quantities?* In this paper we want to discuss this problem from the theoretical point of view, disregarding the restrictions $0 < x_i < 1$ which would add certain inequalities to the conditional equations which we shall obtain for the x_i. It is therefore possible that a solution which is admissible from the algebraical point of view, must be

*We are using the terminology which Professor L. L. Thurstone introduced in his book *The Vectors of Mind* (Chicago, 1935). This work will henceforth be referred to as "*Th*".

discarded by the psychologist if those inequalities are not satisfied.

Since we have only n variables at our disposal, it is clear that the rank of R can certainly not be reduced indefinitely. Indeed, the matrix R possesses minors of order $[\frac{n}{2}]$* (e.g. in the upper right-hand corner) which are independent of the x_i and are assumed to be non-zero, since the r_{ik} do not satisfy any special conditions.
Hence we must have:

$$\varrho \geqslant [\frac{n}{2}] \cdot \tag{1.2}$$

But we shall see that, when $n > 4$, this lower boundary can, in general, not be reached; in fact, it will be proved that

$$\varrho = \{\tfrac{1}{2}(2n + 1 - \sqrt{8n+1})\} , \tag{1.3}$$

where

$\{Z\}$ = smallest integer greater than, or equal to, Z.

The following table contains the values of ϱ corresponding to some small number n of tests:

n	2	3	4	5	6	7	8	9	10	11	12
ϱ	1	1	2	3	3	4	5	6	6	7	8

(1.4)

The expression on the right-hand side of (1.3) also occurs in an inequality connecting the rank ϱ and the number of tests, viz.:

$$\varrho \leqslant \frac{(2n+1) - \sqrt{8n+1}}{2} \tag{1.5}$$

which Professor Thurstone deduced from the general principle that a scientific theory must be "overdetermined" by the data ("*Th.*" p. 76). Mathematically, this means that the data $r_{ik} (i \neq k)$ are supposed to be connected by certain relations so that they *cannot* be regarded as independent variables. In Spearman's classical case of one common factor those relations are the vanishing of all tetrad-differences, but similar conditions should emerge for any number of common factors if Thurstone's principle is applied.

In § § 1-3 we shall treat the coefficients $r_{ik} (i \neq k)$ as independent variables whereas in the last section (§ 4) we shall apply our results to overdetermined problems hoping that, through the mathematical analysis, the precise significance and consequences of

*Following the customary notation: $[Z]$ = largest integer less than or equal to Z.

Professor Thurstone's postulates will be brought out more clearly. We would like to point out, however, that the methods employed are mainly theoretical so that special singularities which are introduced by peculiar numerical constellations, are not taken into account.

I am indebted to Professor Godfrey H. Thomson for suggesting this problem to me and for kindly allowing me to make use of some of his unpublished research notes.

II

The proof of (1.3) rests upon a theorem concerning the rank of a matrix, which is generally attributed to Kronecker. If the matrix (1.1) is of rank ϱ, this means that at least one of its ϱ —rowed minors does not vanish while all its $(\varrho + 1)$ —rowed minors are zero. There are $\binom{n}{\varrho + 1}^2$ such minors and at first sight it might seem as if the n unknown quantities x_i had to satisfy that very large number of conditions. But according to Kronecker's Theorem the number of independent conditions is considerably smaller, because it can be proved that the vanishing of *certain* minors is already sufficient to ensure the rank ϱ of R, i.e. to cause *all* $(\varrho + 1)$—rowed minors to vanish.

We shall apply Kronecker's Theorem* in this form:

The matrix

$$R = \begin{Vmatrix} r_{11} \; r_{12} \cdots\cdots r_{1n} \\ r_{21} \; r_{22} \cdots\cdots r_{2n} \\ \\ r_{n1} \; r_{n2} \cdots\cdots r_{nn} \end{Vmatrix}$$

if of rank ϱ if the ϱ —rowed minor

$$\Delta_0 = \begin{vmatrix} r_{11} \cdots\cdots r_{1\rho} \\ r_{21} \cdots\cdots r_{2\rho} \\ \\ r_{\rho 1} \cdots\cdots r_{\rho\rho} \end{vmatrix} \tag{2.1}$$

is non-zero, whereas all "bordered" minors of

*See M. Bôcher, *Introduction to Higher Algebra*, p. 54 (New York, 1927).

$$\Delta_{\kappa\lambda} = \begin{vmatrix} r_{11} \cdots r_{1\rho} \; r_{1\lambda} \\ r_{\rho 1} \cdots r_{\rho\rho} \; r_{\rho\lambda} \\ r_{\kappa 1} \cdots r_{\kappa\rho} \; r_{\kappa\lambda} \end{vmatrix} \quad (\kappa, \lambda = \varrho + 1, \varrho + 2, \cdots, n) \tag{2.2}$$

vanish.

The number of these bordered determinants is $(n - \varrho)^2$. But since R is a symmetrical matrix, it can be seen by inspection that

$$\Delta_{\kappa\lambda} = \Delta_{\lambda\kappa} \qquad (\kappa, \lambda = \varrho + 1, \varrho + 2, \cdots, n) \, .$$

Hence, the matrix R is of rank ϱ when $\Delta_0 \neq 0$ and

$$\Delta_{\kappa\lambda} = 0 \quad \begin{pmatrix} \kappa, \lambda = \varrho + 1, \varrho + 2, \cdots, n \\ \kappa \leqq \lambda \end{pmatrix} . \tag{2.3}$$

The number of conditions has thus been reduced to

$$k_\rho = \frac{(n - \varrho)(n - \varrho + 1)}{2} \, . \tag{2.4}$$

The number of unknowns is n, namely equal to the number of diagonal elements $x_i = r_{ii}$ in R and the set of conditions can, *in general*, only be satisfied if

$$n \geqq k_\rho \; ,$$

or

$$n \geqq \frac{(n - \varrho)(n - \varrho + 1)}{2}$$

which, after some elementary manipulations, becomes

$$\varrho \geqq \tfrac{1}{2}(2n + 1 - \sqrt{8n + 1}) \, .$$

The smallest possible value for ϱ is therefore

$$\varrho = \{\tfrac{1}{2}(2n + 1) - \tfrac{1}{2}\sqrt{8n + 1}\} \, .$$

The above deduction is, however, still incomplete because it must be shown that the k_ρ equations (2.3) are independent, i.e. that none of them is a consequence of the others. This gap, which, from the mathematical point of view, constitutes the essential part of our proof, will be filled in the next section.

III

In order to show that the conditions (2.3) form a set of k_ρ independent equations for the n unknowns $x_1, x_2, \cdots, x_n$ it is sufficient to prove that they have that property for certain special values of the parameters $r_{ik} (i \neq k)$, because it will then be clear that no identity exists between them.

We shall put

$$r_{\alpha\beta} = 0 \quad \begin{pmatrix} \text{for } \alpha,\beta = 1, 2, \cdots, \varrho \\ \text{and } \alpha \neq \beta \ . \end{pmatrix} \tag{3.1}$$

leaving to a later stage the choice of suitable values for the coefficients r_{ik} in which one suffix is greater than ϱ. The conditions (2.3) then assume the form

$$\Delta_{\kappa\lambda} = \begin{vmatrix} x_1 & 0 & \cdots & 0 & r_{1\lambda} \\ 0 & x_2 & \cdots & 0 & r_{2\lambda} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & x_\rho & r_{\rho\lambda} \\ r_{\kappa 1} & r_{\kappa 2} & \cdots & r_{\kappa\rho} & r_{\kappa\lambda} \end{vmatrix} = 0 \quad \begin{pmatrix} \kappa, \lambda = \varrho + 1, \cdots, n \\ \kappa \leqslant \lambda \end{pmatrix}$$

Expanding the determinant with respect to the last row and the last column we obtain

$$0 = x_1 x_2 \cdots x_\rho r_{\kappa\lambda} - (x_2 x_3 \cdots x_\rho r_{\kappa 1} r_{1\lambda} + x_1 x_3 \cdots x_\rho r_{\kappa 2} r_{2\lambda} + \cdots + x_1 x_2 \cdots x_{\rho-1} r_{\kappa\rho} r_{\rho\lambda})$$

or dividing by $x_1, x_2, \cdots, x_\rho$ and putting

$$z_\sigma = \frac{1}{x_\sigma} \qquad (\sigma = 1, 2, \cdots \ \varrho) \tag{3.2}$$

we can write

$$z_1 r_{\kappa 1} r_{1\lambda} + z_2 r_{\kappa 2} r_{2\lambda} + \cdots + z_\rho r_{\kappa\rho} r_{\rho\lambda} = r_{\kappa\lambda} \quad (\kappa < \lambda), \tag{3.3}$$

$$z_1 r^2_{\kappa 1} + z_2 r^2_{\kappa 2} + \cdots + z_\rho r^2_{\kappa\rho} = x_\kappa \quad (\kappa = \lambda) \ . \tag{3.4}$$

For convenience, we have split up our system of equations into two groups the first of which contains the $\binom{n-\varrho}{2}$ equations belonging to unequal values of κ and λ while the $(n - \varrho)$ equations of the second group correspond to the cases in which $\kappa = \lambda$. The first group involves

only the first ϱ unknowns in terms of which the remaining $(n-\varrho)$ unknowns can be expressed in virtue of (3.4). It is therefore sufficient to consider the first group of equations only thus reducing both the number of equations and unknowns by the same amount, viz, by $n-\varrho$.

We have now to investigate whether the $\binom{n-\varrho}{2}$ equations are independent with respect to the ϱ unknowns $z_1, z_2, \cdots, z_\rho$. Since the equations are linear in the ϱ unknowns, we have only to show that the matrix of the coefficients is of rank ϱ if $\varrho \leqslant \binom{n-\varrho}{2}$ and of rank $\binom{n-\varrho}{2}$, if $\varrho \geqslant \binom{n-\varrho}{2}$. Again, it is only necessary to prove this for special values of the coefficients $r_{\kappa\sigma}$ $(\kappa=\varrho, \varrho+1, \cdots, n;\ \sigma=1, 2, \cdots, \varrho)$. Let

$$u_1, u_2, \cdots, u_\rho \tag{3.5}$$

be ϱ arbitrary quantities and let $n-\varrho$ constants

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-\rho}$$

be chosen such that

$$\varepsilon_\kappa + \varepsilon_\lambda \neq \varepsilon_{\kappa'} + \varepsilon_{\lambda'} \begin{pmatrix} \kappa, \kappa', \lambda, \lambda' = \varrho+1, \varrho+2, \cdots, n \\ \kappa < \lambda\ ;\quad \kappa' < \lambda' \end{pmatrix}$$

unless $\kappa=\kappa'$ and $\lambda=\lambda'$. (This condition is satisfied if we put

$$\varepsilon_\kappa = 2^\kappa ,$$

because every number can be written as a sum of powers of 2 in only one way). We now fix a definite order for the pairs of numbers (κ, λ) and denote the $\binom{n-\varrho}{2}$ distinct quantities

$$\varepsilon_\kappa + \varepsilon_\lambda$$

by

$$\omega_1, \omega_2, \cdots, \omega_m ,$$

where, for brevity, we write

$$m = \binom{n-\varrho}{2}.$$

If we now put

$$r_{\kappa\sigma} = r_{\sigma\kappa} = u_\sigma^{\varepsilon_\kappa} \sigma = 1, 2, \cdots, \varrho;\ \kappa = \varrho+1, \varrho+2, \cdots, n) , \tag{3.6}$$

the equations (3.3) assume the form

$$u_1^{\omega_\mu} z_1 + u_2^{\omega_\mu} z_2 + \cdots + u_\rho^{\omega_\mu} z_\rho = r_{\kappa\lambda} \; (u = 1, 2, \cdots, m) \; . \tag{3.7}$$

The $(m \times \varrho)$—matrix whose rank we have to investigate, is

$$\left| \begin{matrix} u_1^{\omega_1} & u_2^{\omega_1} & \cdots & u_\rho^{\omega_1} \\ u_1^{\omega_2} & u_2^{\omega_2} & \cdots & u_\rho^{\omega_2} \\ u_1^{\omega_m} & u_2^{\omega_m} & \cdots & u_\rho^{\omega_m} \end{matrix} \right| \; . \tag{3.8}$$

Our proposition will certainly be proved if we can show that every minor of the matrix (3.8) is non-zero. A typical minor of (3.8) may be denoted by

$$\left| \begin{matrix} u_a^{\omega_\alpha} & u_b^{\omega_\alpha} & \cdots & u_h^{\omega_\alpha} \\ u_a^{\omega_\beta} & u_b^{\omega_\beta} & \cdots & u_h^{\omega_\beta} \\ u_a^{\omega_\eta} & u_b^{\omega_\eta} & \cdots & u_h^{\omega_\eta} \end{matrix} \right| \; . \tag{3.9}$$

But it is known that a determinant of the form (3.9) does not vanish identically provided that the exponents ω_α, ω_β, $\cdots$ are distinct. Hence the equations (3.7) or (3.3) are independent. The argument at the end of section II was therefore justified and formula (1.3) is proved.

IV

We shall now draw some conclusions from our results and discuss their relations to Professor Thurstone's inequality (1.5) and the principle of overdeterminacy. Suppose that the actual rank of a correlational matrix — after choosing suitable communalities — is equal to r. In order that this rank might be attained it is necessary to solve.

$$k_r = \tfrac{1}{2}(n - r)(n - r + 1)$$

equations for n unknowns. With arbitrary coefficients $r_{ik} (i \neq k)$; this is possible if and only if the number

$$f_r = n - k_r = \tfrac{1}{2}[-n^2 + n(2r + 1) - (r^2 - r)] \tag{4.1}$$

is non-negative $(\geqslant 0)$. If $f_r = 0$, we have as many equations as unknowns and a finite number of solutions is to be expected; if $f_r > 0$,

there will be an infinity of solutions the general solution having f_r "degrees of freedom", i.e. involving arbitrary parameters. On the other hand, if $f_r = -g_r < 0$, no solution can be obtained unless the coefficients r_{ik} satisfy at least g_r relations on account of which the number of independent conditions (2.3) for the x_i is reduced to n. It is this case which, according to Thurstone, should always occur.

The following table shows the number of degrees of freedom or the number of conditions on the r_{ik} for small values of n and r

n	3	4	5	6	7
f_1	0	—2	—5	—9	—14
f_2	2	1	—1	—4	—8
f_3	3	3	2	0	—3

E.g. Three tests can always be reduced to rank 1 in a finite number of ways (namely in one way only, if certain inequalities are fulfilled). If 4 tests are to be reduced to rank I, the correlation coefficients must satisfy 2 conditions which may be written

$$r_{12}\, r_{34} - r_{13}\, r_{24} = 0 \ ,$$

$$r_{14}\, r_{23} - r_{12}\, r_{34} = 0 \ .$$

As a consequence of these, the third tetrad difference

$$r_{14}\, r_{23} - r_{13}\, r_{24}$$

will also vanish being the sum of the former two.

Four tests can always be reduced to rank 2 in an infinite number of ways the general solution having one degree of freedom. In five variables such a reduction is impossible unless the correlations r_{ik} satisfy one condition (see below). Finally, we mention that with 6 tests it is in general possible to attain rank 3 without any restrictions on the coefficients r_{ik}.

The question now arises: What are those relations which the r_{ik} must satisfy when the problem is overdetermined, i.e. when the rank is lower than it is to be expected? It appears that they are rather complicated functions which may perhaps be of little practical value; but it would certainly be of some theoretical interest if an explicit formula for those functions could be obtained. We will leave this for further investigation. For the case $n = 5$, $r = 2$ we have

found that the (one) condition is:

$$0 = r_{12}\,r_{23}\,r_{34}\,r_{45}\,r_{51} - r_{12}\,r_{23}\,r_{35}\,r_{41}\,r_{54} - r_{12}\,r_{24}\,r_{35}\,r_{43}\,r_{51}$$
$$+ r_{12}\,r_{24}\,r_{31}\,r_{45}\,r_{53} + r_{12}\,r_{25}\,r_{34}\,r_{41}\,r_{53} - r_{12}\,r_{25}\,r_{31}\,r_{43}\,r_{54}$$
$$- r_{13}\,r_{24}\,r_{35}\,r_{41}\,r_{52} + r_{13}\,r_{25}\,r_{34}\,r_{42}\,r_{51} + r_{14}\,r_{23}\,r_{31}\,r_{45}\,r_{52}$$
$$- r_{14}\,r_{25}\,r_{32}\,r_{43}\,r_{51} - r_{15}\,r_{23}\,r_{31}\,r_{42}\,r_{54} + r_{15}\,r_{24}\,r_{32}\,r_{41}\,r_{53}\ .^{*}$$

*My attention has been drawn to Professor T. L. Kelley's *pentad criterion* which is the same as the formula given (see *Crossroads in the Mind of Man,* page 58). As my method of arriving at it is shorter than his perhaps it is worth while giving it.

If 5 communalities can be found such that the matrix

$$\begin{vmatrix} x_1 & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & x_2 & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & x_3 & r_{34} & r_{35} \\ r_{41} & r_{42} & r_{43} & x_4 & r_{45} \\ r_{51} & r_{52} & r_{53} & r_{54} & x_5 \end{vmatrix}$$

is *of* rank 2, then *all* 3-rowed minors must vanish; in particular, we must have

$$\begin{vmatrix} x_1 & r_{12} & r_{13} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \end{vmatrix} = 0 \text{ and} \begin{vmatrix} x_1 & r_{12} & r_{14} \\ r_{31} & r_{32} & r_{34} \\ r_{51} & r_{52} & r_{54} \end{vmatrix} = 0\ .$$

These are two linear equations for x_1, viz.:

$$x_1(r_{42}\,r_{53} - r_{43}\,r_{52}) = r_{13}\,r_{42}\,r_{51}$$
$$+ r_{12}\,r_{41}\,r_{53} - r_{12}\,r_{43}\,r_{51} - r_{13}\,r_{41}\,r_{52}\ ,$$
$$x_1(r_{32}\,r_{54} - r_{34}\,r_{52}) = r_{14}\,r_{32}\,r_{51}$$
$$+ r_{12}\,r_{31}\,r_{54} - r_{12}\,r_{34}\,r_{51} - r_{14}\,r_{31}\,r_{52}\ .$$

It will be noticed that the second equation can be derived from the first equation by interchanging the suffixes 3 and 4. On eliminating x_1 we obtain the required equation.

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THE SIMULTANEOUS PREDICTION OF ANY NUMBER OF CRITERIA BY THE USE OF A UNIQUE SET OF WEIGHTS

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When several criteria are available, it is ordinarily necessary (1) to select one of them as *the* criterion, (2) to use several and thus arrive at several different sets of weights, or (3) to combine them into a single measure. A formula is derived for the determination of a unique set of weights. The use of these weights will produce the highest possible average coefficient of correlation between the various criteria and two (or more) weighted independent variables. If desired, the criteria may be assigned any predetermined weights. The weights then derived for the independent variables are such that the weighted average of the correlation coefficients between the various criteria and the independent variable composite will be a maximum. In the use of these formulas, no assumptions are necessary regarding the interrelationships existing among the criteria and it is not necessary to compute the intercorrelations among the criteria. A numerical example is included.

Often investigations are made in which several variables (such as measures of various aspects of a given ability) are available which may be used as criteria for a determination of the weights to be assigned to two or more independent variables. The question of which measure to select as *the* criterion has sometimes assumed considerable importance. Some investigators have avoided the difficulties involved in the selection of a criterion by using several such measures and allowing the reader to choose among them. However, even though several criteria may be available, it is seldom feasible to use more than one set of weights for prediction purposes.

As an illustration, one might wish to weight a personality test and an interest test in such a manner that the weighted composite would have as high a correlation as possible with the subsequent success of a group of new life insurance agents. If this criterion of subsequent success were uniquely defined and universally agreed upon, there would be no ambiguity and the usual multiple correlation solution would yield the required weights. In actual practice, however, one finds that the "subsequent success of life insurance agents" may be measured objectively in at least the following ways:

(1-4) total amount of insurance sold during first 3, 6, 12, or 24 months;

(5-8) total amount of premiums on policies sold during first 3, 6, 12, or 24 months;

(9-12) total amount of commissions received during first 3, 6, 12, or 24 months;

(13-16) present and future potentialities, as rated by the supervisor or manager at the end of 3, 6, 12, or 24 months; or even by

(17-20) the dichotomous variable of whether or not the agent is still in the life insurance business at the end of 3, 6, 12, or 24 months.

In any given problem, it is quite likely that a good many of these twenty possible criteria can be eliminated, but it is also quite likely that it would prove desirable to retain more than one of them. It is also desirable that the weighted composite of the independent variables should correlate highly with each of the criteria that are retained. What is needed, therefore, is a method whereby the independent variables may be weighted in such a manner that the average of the correlations between the various criteria and the independent variable composite shall be as high as possible.

The formulas developed in this article were derived in order to make possible the determination of weights to be assigned the independent variables while simultaneously using several criteria instead of limiting the investigator to only one criterion measure. If desired, the criterion measures may be weighted in any suitable manner.

With most methods of combining criteria, the assumption is implicitly made that the various criteria are measuring somewhat the same thing. Note that the present method (which does not combine the criteria) makes no such assumption and is entirely independent of and uninfluenced by the intercorrelations among the criteria. For example, if one should attempt to measure the effectiveness of a combination of two or more intelligence, achievement, attitude, or personality tests in predicting "success in life ten years later", he might choose as his criteria such variables as (1) an index of prominence in the community, (2) an index measuring professional reputation, (3) a measure of happiness and satisfaction with present status, and (4) annual income at the end of the ten years. Situations can easily be conceived in which such variables as these would have low (or even

zero or negative) intercorrelations. These variables might, nevertheless, be regarded as measures of different aspects of the criterion defined above. Even under circumstances as extreme as these, the formulae which are derived below will still yield definite, unique, and meaningful results.

In order to simplify the following derivation, a weight of 1.00 is arbitrarily assigned to the first independent variable. The problem then becomes one of determining an appropriate weight, b, which will maximize the average correlation between the several criterion measures and the independent variable composite, $x_1 + bx_2$.

By a common formula,

$$r_{(y)(x_1+bx_2)} = \frac{\Sigma(y)(x_1+bx_2)}{\sqrt{\Sigma y^2}\sqrt{\Sigma(x_1+bx_2)^2}} \; .$$

If standard scores are used, this reduces immediately to

$$r_{(y)(x_1+bx_2)} = \frac{r_{yx_1} + br_{yx_2}}{\sqrt{1+2br_{x_1x_2}+b^2}} \; * \qquad (1)$$

The average of the n coefficients of correlation between the independent variable composite, $x_1 + bx_2$, and the n criterion measures, $y_3, y_4, \cdots, y_{n+2}$, is, by definition,

$$\bar{r}_{(y_i)(x_1+bx_2)} = \frac{1}{n}\left[r_{(y_3)(x_1+bx_2)} + r_{(y_4)(x_1+bx_2)} + \cdots + r_{(y_{n+2})(x_1+bx_2)}\right] \; .$$

In view of (1), this may be written

$$\bar{r}_{(y_i)(x_1+bx_2)} = \frac{(r_{y_3x_1} + r_{y_4x_1} + \cdots + r_{y_{n+2}x_1}) + b(r_{y_3x_2} + r_{y_4x_2} + \cdots + r_{y_{n+2}x_2})}{n\sqrt{1+2br_{x_1x_2}+b^2}}$$

$$= \frac{\Sigma r_{y_ix_1} + b\Sigma r_{y_ix_2}}{n\sqrt{1+2br_{x_1x_2}+b^2}} \qquad (i = 3, 4, \cdots, {}_{n+2}) \qquad (2)$$

By logarithmic differentiation,

$$\frac{\dfrac{d\bar{r}_{(y_i)(x_1+bx_2)}}{db}}{\bar{r}_{(y_i)(x_1+bx_2)}} = \frac{\Sigma r_{y_ix_2}}{\Sigma r_{y_ix_1} + b\Sigma r_{y_ix_2}} - \frac{1}{2}\,\frac{2r_{x_1x_2}+2b}{1+2br_{x_1x_2}+b^2}$$

*Except for changes in notation, this is the same formula as that given by Thurstone, L. L. (A scoring method for mental tests. *Psychol. Bull.*, 1919, **16**, no. 7.) and by Kelley, T. L. (*Statistical method.* New York: The Macmillan Company, 1924. Formula 150.)

which, after setting the derivative equal to zero and solving for b, gives

$$b = \frac{\Sigma r_{y_i x_2} - r_{x_1 x_2} \Sigma r_{y_i x_1}}{\Sigma r_{y_i x_1} - r_{x_1 x_2} \Sigma r_{y_i x_2}} \tag{3}$$

This completes the solution for the case in which it is desired to assign equal weight or importance to each of the criteria. When x_1 is assigned a weight of 1.00 and x_2 is weighted in accordance with (3), the average correlation between the weighted independent variable composite, $x_1 + bx_2$ and the various criteria will be a maximum.* If desired, the individual correlation coefficients between any one or each one of the criteria and the independent variable composite may be computed by (1). If these correlation coefficients are averaged, the same value as that given by (2) will, of course, be obtained.

Perhaps a different approach to this problem will serve to show more clearly the meaning of formulas (2) and (3). Let us first assume an arbitrary weight, $b = 5$, for x_2, and calculate the individual correlations between the independent variable composite, $x_1 + 5x_2$, and each of the criteria, $y_3, y_4, \cdots, y_{n+2}$. Also compute the arithmetic mean of these criterion correlation coefficients.

Let us then assume a second arbitrary weight, $b = -.03$, for x_2, and calculate the individual correlations between the independent variable composite, $x_1 - .03x_2$, and each of the criteria. Also compute the average criterion correlation coefficient.

Assume that this process were continued until all of the infinite number of possible weights had been tried out. If then the infinite number of average criterion correlation coefficients were examined,

*One minor exception should be mentioned. In assigning a weight of 1.00 to x_1, we are automatically assuming that x_1 should be weighted positively. If a higher correlation would be obtained by weighting x_1 negatively, the obtained value of $\overline{r}_{(y_i)(x_1+bx_2)}$ will turn out to be negative, but equal in absolute value to the maximal value. The correct weights and correlation coefficients may then be obtained by changing the weight of x_1 to -1.00 and reversing the signs of the values already obtained for b and $\overline{r}_{(y_i)(x_1+bx_2)}$.

Since, as in the exceptions just cited, $\overline{r}_{(y_i)(x_1+bx_2)}$ is sometimes a minimum (in algebraic value) rather than a maximum, a question arises as to whether it is a maximum in the ordinary case in which both x_1 and x_2 are weighted positively. The writer has not proved directly that such a value of $\overline{r}_{(y_i)(x_1+bx_2)}$ is a maximum, but the inference seems clear, because it can be proved that under such conditions another set of weights for x_1 and x_2 (directly proportional to 1.00 and b) will give a minimum standard error of estimate.

it would be found that one of them had the unique characteristic of being larger than any of the others, and special interest would, of course, attach to it and to the weight, b, which had been used in obtaining it.

Fortunately, it is not necessary to make an infinite, or even a large, number of calculations in order to determine these two values. The weight, b, that should be used to obtain the highest average criterion correlation is given by equation (3) and the corresponding maximal correlation is given by equation (2).

If the various criterion measures are not all judged to be equally important, or if they are unequally reliable, the investigator may not wish to attach equal significance to each of them, as has been assumed in the preceding formulas. The solution of the more general case in which any predetermined set of weights, w_i, is assigned to the criteria proceeds along lines strictly analogous to those followed above, giving

$$\bar{r}_{(wy_i)(x_1+bx_2)} = \frac{\Sigma w_i r_{y_i x_1} + b\Sigma w_i r_{y_i x_2}}{\Sigma w_i \sqrt{1 + 2br_{x_1 x_2} + b^2}} \qquad (4)$$

in which b is defined by

$$b = \frac{\Sigma w_i r_{y_i x_2} - r_{x_1 x_2}\Sigma w_i r_{y_i x_1}}{\Sigma w_i r_{y_i x_1} - r_{x_1 x_2}\Sigma w_i r_{y_i x_2}} \qquad (5)$$

It should be mentioned that formulas (2) and (4) are general formulas in that they will give the average of the n correlations between the criterion measures and *any* weighted combination of the independent variables. Thus, if one desires to do so, any value whatever may be substituted for b and the resulting average correlation will be obtained. When the value of b given by (3) is substituted in (2), $\bar{r}_{(y_i)(x_1+bx_2)}$ will, of course, take its maximum value. Similarly, when the value of b given by (5) is substituted in (4), $\bar{r}_{(wy_i)(x_1+bx_2)}$ will take its maximum value.

An example may serve to illustrate the use of this method. We are given the following table of intercorrelations and the information that weights of 2, 3, 6, and 5 are to be assigned to the variables y_3, y_4, y_5, and y_6 respectively.

	y_3	y_4	y_5	y_6	x_1	x_2
y_3	1.00				.37	—.32
y_4		1.00			.33	—.34
y_5			1.00		.28	—.51
y_6				1.00	.29	—.25
x_1	.37	.33	.28	.29	1.00	—.24
x_2	—.32	—.34	—.51	—.25	—.24	1.00

Then $\Sigma w_i r_{y_i x_1} = 2(.37) + 3(.33) + 6(.28) + 5(.29) = 4.86$

and $\Sigma w_i r_{y_i x_2} = 2(-.32) + 3(-.34) + 6(-.51) + 5(-.25) = -5.97$

Hence, by formula (5), b

$$= \frac{-5.97 - (-.24)\,4.86}{4.86 + (-.24)\,5.97} = \frac{-4.80}{3.43} = -1.40$$

and, by formula (4), $\bar{r}_{(wy_i)(x_1+bx_2)}$

$$= \frac{4.86 - 1.40(-5.97)}{16\sqrt{1 + 2(-1.40)(-.24) + (-1.40)^2}} = \frac{13.22}{30.50} = .43\ .$$

The writer has not yet completed the solution for the more general case in which any number of independent variables may be used. At the present time any one of a number of approximation procedures may be used, one of the simplest of which is to assign new weights $\frac{1}{1+b}$ and $\frac{b}{1+b}$ to the first two independent variables in order that their sum shall be equal to 1.00. Then the sum of these two weighted variables will be treated as a single variable, x'_1, having the correlations with the criteria (and with the third independent variable) given by (1). The third independent variable, x'_2, is now introduced and then x'_1 and x'_2 are treated just as x_1 and x_2 were previously treated. This procedure may then be extended to a fourth variable, x''_2, by letting $x''_1 = x'_1 + bx'_2$, reassigning proportional weights to the variables, and continuing the process. Ordinarily, the independent variables having the higher average criterion correlations should be introduced first.

It may be of interest to compare the method presented in this article with some of the other methods recently developed for dealing with problems in which several criteria are available.

Hotelling* has solved "the problem of finding a linear function of the criterion variates which can most accurately be predicted from given observations, in the sense of least squares." Hotelling states, however, that "it is entirely possible that the most predictable criterion thus defined will have little correlation with many of the original criteria, especially if they are numerous," and he gives a suggestion for increasing the correlation with the first few principal components.

Horst† developed both an exact and an approximate method of combining the various criteria into a single measure on the assumption that "the separate measures should be combined in such a manner that the composite measure will result in giving the maximum difference between all possible pairs of members in the group."

Edgerton and Kolbe‡ have derived a method for determining weights for each of n criteria in such a manner as to make the unweighted differences among the n standard scores for an individual a minimum. They show that the weights obtained by their method are the same as those obtained by Horst's exact solution.

In most formulas for correlations between sums or averages of statistical measures, it is necessary to compute all of the possible intercorrelations. Formulas (2) and (4), however, are not correlations between averages, but are averages of correlations; and their solution does not necessitate the computation of any of the intercorrelations among the criteria. With the two independent and n criterion measures used here, unless the intercorrelations among the criteria are wanted for other purposes, only $(2n + 1)$ correlations need be solved instead of the $\frac{(n+2)(n+1)}{2}$ possible correlations. With a small number of criteria, this saving is negligible, but when the number of criteria is greater than five, less than half of the possible correlations among all the variables are needed if formula (2) or (4) is used. For instance, if in the illustration relating to life insurance agents, ten, or half, of the criteria were retained, only 21 of the 66 possible zero order correlation coefficients would need to be computed.

*Hotelling, Harold: "The most predictable criterion." *Journal of Educational Psychology*, February, 1935, **26**, 139-142.

†Horst, Paul: "Obtaining a composite measure from a number of different measures of the same attribute," *Psychometrika*, March, 1936, **1**, 53-60.

‡Edgerton, Harold A. and Kolbe, Laverne E.: "The method of minimum variation for the combination of criteria," *Psychometrika*, September, 1936, **1**, 183-187.

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NOTE ON EXCITATION THEORIES

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Offner's demonstration of the excitation time equivalence, for all currents, of the theories of Hill and Rashevsky is here extended to a more general case which includes both. It is found that any system of this general type may be replaced by an equivalent one of the Rashevsky type, thus effecting considerable simplification in the mathematical detail.

It has been shown by Offner[1] that the excitation equations of Rashevsky [2]

$$\left.\begin{aligned} \frac{de}{dt} &= K\,I(t) - k(e - e_0) \\ \frac{di}{dt} &= M\,I(t) - m(i - i_0) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (1)$$

and those of Hill[3]

$$\left.\begin{aligned} \frac{de'}{dt} &= K'I(t) - k'(e' - e_0') \\ \frac{di'}{dt} &= M'(e' - e_0') - m'(i' - i_0') \end{aligned}\right\} \quad \text{- - - - -} \quad (2)$$

will give the same values for the excitation time (at which $e = i$) for all $I(t)$, if the constants be related by

$$\left.\begin{aligned} k' &= k \\ m' &= m \\ K' &= r(K - M) \\ M' &= M(k - m)/(K - M) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (3)$$

The zero subscripts denote values at $t = 0$, and $r = \dfrac{(e_0' - i_0')}{(e_0 - i_0)}$.

Both (1) and (2) are included under

*The author is indebted to Mr. Alvin Weinberg of the Dept. of Physics, University of Chicago, for verifying the computations.

(1) Unpublished work of Mr. E. A. Offner, Dept. of Physiology, University of Chicago.

(2) Rashevsky, N., 1933. *Protoplasma,* V. 20, 42.

(3) Hill, A. V., 1936. *Proc. Roy. Soc. London,* B, V. 119, 305. Here given in different notation.

$$\left.\begin{aligned} \frac{de}{dt} &= k_{11}(e - e_0) + k_{12}(i - i_0) + a\,I \\ \frac{di}{dt} &= k_{21}(e - e_0) + k_{22}(i - i_0) + a\,b\,I\,, \end{aligned}\right\} \text{- - - - - -} \quad (4)$$

and in view of the above result it seems of interest to consider this more general case.

Solution of Equations: The solution of a system of simultaneous linear differential equations with constant coefficients can always be reduced to quadratures. This may be done in a number of ways, but perhaps the most convenient is in a notation due to Bartky.[4]

The system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= k_{11}x_1 + \cdots + k_{1n}x_n + \varphi_1(t) \\ &\vdots \\ \frac{dx_n}{dt} &= k_{n1}x_1 + \cdots + k_{nn}x_n + \varphi_n(t) \end{aligned}\right\} , \text{- - - - - -} \quad (5)$$

or, in vector-matrix notation,

$$\frac{dX}{dt} = K\,X + \varphi(t) \ ,$$

has the general solution

$$X = e^{Kt}X_0 + e^{Kt}\int_0^t e^{-Kt}\,\varphi\,dt \ ; \qquad \text{- - - -} \quad (6)$$

where X_0 is the vector of initial values, and e^{Kt} is a matrix which, for a 2×2 matrix K, is defined as follows:

$$\begin{aligned} e^{Kt} &= \frac{1}{\lambda_1 - \lambda_2}\,e^{\lambda_1 t}(K - \lambda_2 E) + \frac{1}{\lambda_2 - \lambda_1}\,e^{\lambda_2 t}(K - \lambda_1 E) \ \text{if } \lambda_1 \neq \lambda_2 \ , \\ e^{Kt} &= e^{\lambda t}[K - (\lambda - 1)E] \ \text{if } \lambda_1 = \lambda_2 = \lambda \ ; \end{aligned} \quad (7)$$

where λ_1, λ_2 are the latent roots of K, and E is the identity matrix.

For the case $n = 2$, $\lambda_1 \neq \lambda_2$, the vector integral in (6) has components Ψ_1, Ψ_2 given by

$$\left.\begin{aligned} (\lambda_1 - \lambda_2)\,\Psi_1 &= (k_{11} - \lambda_2)I_{11} + k_{12}I_{21} - (k_{11} - \lambda_1)I_{12} - k_{12}I_{22} \\ (\lambda_1 - \lambda_2)\,\Psi_2 &= k_{21}I_{11} + (k_{22} - \lambda_2)I_{21} - k_{21}I_{12} - (k_{22} - \lambda_1)I_{22} \end{aligned}\right\} \text{-} (8)$$

[4] MacMillan, W. D., *Dynamics of Rigid Bodies*, pp. 413-434.

where

$$I_{ij}(t) = \int_0^t \varphi_i e^{-\lambda_j t} dt \quad - - - - - - - \quad (9)$$

The first component of $\vartheta = e^{Kt}\Psi$ is

$$\vartheta_1 = \left| \begin{array}{l} \{(k_{11} - \lambda_2)^2 + k_{12}k_{21}\} F_{111} \\ + k_{12}(k_{11} + k_{22} - 2\lambda_2) F_{121} \\ - \{(k_{11} - \lambda_2)(k_{11} - \lambda_1) + k_{12}k_{21}\}(F_{211} + F_{112}) \\ - k_{12}(k_{11} + k_{22} - \lambda_1 - \lambda_2)(F_{221} + F_{122}) \\ + \{(k_{11} - \lambda_1)^2 + k_{12}k_{21}\} F_{212} \\ + k_{12}(k_{11} + k_{22} - 2\lambda_1) F_{222} \end{array} \right| \cdot \frac{1}{(\lambda_1 - \lambda_2)^2} \quad - - - \quad (10)$$

where

$$F_{nij} = e^{\lambda_n t} I_{ij} \,. \quad - - - - - - - - - \quad (11)$$

Using the fact that λ_1, λ_2 satisfy

$$(k_{11} - \lambda)(k_{22} - \lambda) = k_{12}k_{21} \quad (\text{so that } \lambda_1 + \lambda_2 = k_{11} + k_{22}) \,,$$

this simplifies to

$$\vartheta_1 = \{(k_{11} - \lambda_2)F_{111} + k_{12}F_{121} - (k_{11} - \lambda_1)F_{212} - k_{12}F_{222}\}/(\lambda_1 - \lambda_2) \quad - - - - \quad (12)$$

The second component, obtained by interchanging the subscripts, is

$$\vartheta_2 = \{k_{21}F_{111} + (k_{22} - \lambda_2)F_{121} - k_{21}F_{212} - (k_{22} - \lambda_1) F_{222}\}/(\lambda_1 - \lambda_2) \,. \quad - - - - - - - - \quad (13)$$

Application to Excitation Theory: In (4) take $x_1 = e - e_0$, $x_2 = i - i_0$. Since now $\varphi_2 = b\,\varphi_1$ it follows from (9) and (11) that $F_{n2j} = bF_{n1j}$, and (12), (13) reduce to

$$\vartheta_1 = \{(k_{11} - \lambda_2 + bk_{12})F_{111} - (k_{11} - \lambda_1 + bk_{12})F_{212}\}/(\lambda_1 - \lambda_2) \quad - - - - - - - \quad (14)$$

$$\vartheta_2 = [\{k_{21} + b(k_{22} - \lambda_2)\}F_{111} - \{k_{21} + b(k_{22} - \lambda_1)\}F_{212}]/(\lambda_1 - \lambda_2) \,. \quad - - - - - - - - \quad (15)$$

Since the initial conditions are $x_1 = x_2 = 0$ at $t = 0$, the first term on the right of (6) is absent; hence x_1 and x_2 are given by (14) and (15) respectively.

The excitation time is to be determined by setting $e = i$; that is, $x_1 - x_2 = i_0 - e_0$, or

$$\{k_{11} - \lambda_2 - k_{21} + b(k_{12} - k_{22} + \lambda_2)\}\, a\, J_{11} - \{k_{11} - \lambda_1 - k_{21} + b(k_{12} - k_{22} + \lambda_1)\}\, a\, J_{22} = (\lambda_1 - \lambda_2)(i_0 - e_0) \quad - - - - - - - - - \quad (16)$$

where

$$J_{ij}(t) = e^{\lambda_i t} \int_0^t I(t) e^{-\lambda_j t} dt \ .$$

Denoting the respective brackets by A and B, this may be written

$$a\, A\, J_{11} - a\, B\, J_{22} = (\lambda_1 - \lambda_2)\,(i_0 - e_0) \ .$$

In connection with (16) consider the corresponding equation obtained with different (primed) constants in (4). If the corresponding λ_i are equal, then $J_{ij}' \equiv J_{ij}$. The two equations will then be identical if

$$\left.\begin{aligned} r\, a\, A &\equiv a'\, A' \\ r\, a\, B &\equiv a'\, B' \end{aligned}\right\} \quad \text{- - - - - -} \quad (17)$$

Subtracting, $r\, a\,(A - B) = a'(A' - B')$ or $r\, a(1 - b) = a'(1 - b')$. Using this in (17) gives

$$r\, a\{k_{11} - k_{21} + b\,(k_{12} - k_{22})\} = a'\{k'_{11} - k'_{21} + b'(k'_{12} - k'_{22})\} \ ;$$

$$\text{- - - - - - -} \quad (18)$$

which, with

$$\left.\begin{aligned} \lambda_1 &= \lambda_1' \\ \lambda_2 &= \lambda_2' \\ a'(1 - b') &= ra(1 - b) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (19)$$

constitutes four relations on the six quantities k_{ij}, a, b, sufficient to make (16) identical for the primed and unprimed systems. These relations correspond to the equations (3) in the special case there considered, but are less restrictive since the latent roots in (19) may be numbered in either order. Thus Offner's case has the solution

$$\left.\begin{aligned} k' &= m \\ m' &= k \\ K' &= r(K - M) \\ M' &= K(k - m)/(K - M) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (20)$$

as well as that, (3), given by him.

The above results show that any system of type (4) can be replaced, at least insofar as regards excitation time, by an equivalent system of type (1) (or (2)). Thus the fit to observation can not be improved by passing to the more complicated case (4); but, by allowing for a greater variety of mechanism, this may provide a more flexible physical interpretation of the values of the constants in (1).

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STUDIES IN THE LEARNING FUNCTION*

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From Thurstone's theoretical learning curve, a solution for the difficulty of the problem and the learning constant of the subject has been developed. The curve is an equilateral hyperbola. Therefore the semi-major axis represents the learning situation in one constant. The vertex of the curve is a point where all of the animals are equated, since they are all making errors at the rate of one error per trial.

II. Critical Values of the Learning Curve.

In a recent paper we have shown that the theoretical learning curve developed by L. L. Thurstone ('30, '33) fits learning data collected from the Lashley-type maze. In this study we have developed a solution in which it is possible to separate the learning ability of the animal and the difficulty of the problem. We have also examined the critical values of the curve and their merits as criteria of learning.

For this analysis, the learning records from the previous paper were used. The subjects were thirty rats, with brain lesions, trained on Maze I and on Maze V. Maze I is a 4 cul-de-sac maze and V is an 8 cul-de-sac maze. A full description of the animals and training procedure appears in Lashley and Wiley, "Mass Action in Relation to the Number of Elements in the Problem to be Learned." ('33)

The theoretical learning curve as developed by Thurstone appears in the form:

$$u = \frac{\sqrt{m}}{aK} - \frac{\sqrt{m}}{K} \cdot \frac{u}{R}, \tag{1}$$

where u represents the accumulated errors, R is the number of trials, K the learning constant of the animal, m the difficulty of the maze, and a is an arbitrary constant that can be absorbed in m or in the unit of measurement. The origin of the system of coordinates is at the point where *learning* begins.

*A grant-in-aid from the National Research Council has made the present analysis possible. We wish to express our appreciation, for the aid and encouragement, of Professor Rufus Crane of the mathematics department, Ohio Wesleyan University and of Professor L. L. Thurstone, University of Chicago.

In our analysis the origin lies necessarily at the point where *training* begins and we do not know beforehand just when learning begins. The curve therefore appears in the form:

$$u' = A + \frac{BR'}{C + R'}\,, \tag{2}$$

in which u' and R' are the accumulated errors and trials measured from the point at which training begins, and A, B, and C are constants.

The relationship between the two systems of coordinates is expressed by

$$u' = u + e\ ,\ R' = R + r\ , \tag{3}$$

in which e represents the number of errors the rat makes before he begins to learn and r represents the number of trials he must run before he begins to learn. In the previous paper we have shown how (2) may be derived from (1).

The equations for the asymptotes were also derived. The horizontal asymptotes correlated highly with the former criterion of learning, total-errors-minus-errors-on-the-first-trial. A linear relationship exists between the horizontal asymptote on Maze I and the corresponding asymptote on Maze V, which establishes the fact that what a rat does on one maze he tends to do on another of the same type but different difficulty.

In considering the asymptotes of the curve as learning criteria it is necessary to remember that they are measured from the place where training begins. The initial adjustment period, which is not a function of learning alone, is included to some extent. To what extent the adjustment period is included we do not know. All of these animals were operated upon before training and they had been handled extensively. Some of them may have done a great deal of adjusting before they were put in the maze.

Perhaps in discussing the effects of cortical insult upon the ability of the rat to accomplish a particular problem we should include that part of the training record that appears before learning begins. It may be that the removal of part of the cortex may influence the initial adjustment period. On the other hand if we are dealing solely with the influence of the lesion on the ability of the animal to *learn* it must be excluded. With this in mind we have attempted to find some value from which the learning constant of the animal can be derived without including the values of e and r.

In dealing with any conic the value of the eccentricity of the conic is important since it defines the curve as ellipse, parabola, or hyperbola and since it determines the curvature. A conic is defined as the path of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line. The ratio is the eccentricity. In the case of the hyperbola it is the ratio of *the distance from the center of the conic to the focus* to *the distance from the center to the vertex.* It is always greater than 1.00. The value of the eccentricity is the only single constant that will distinguish one conic from another.

In order to determine the value of the eccentricity of equation (2) it is necessary to convert it to the standard form in which the intersection of the two asymptotes is the origin and the asymptotes themselves are the axes of ordinates.

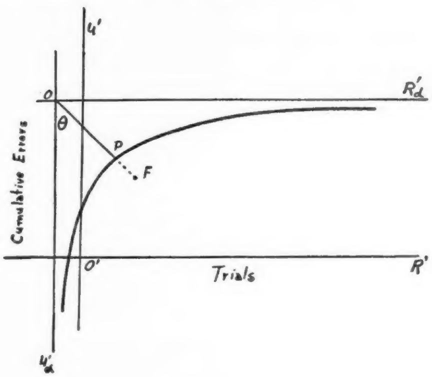

Figure 1. The theoretical learning curve

$$u = A + \frac{BR}{C + R}.$$

The accumulated errors are represented on $O'u'$, the trials on $O'R'$, O' being the point where training begins. P is the vertex of the curve. OR'_a is the horizontal asymptote and Ou'_a is the vertical asymptote. O is the point at which the two asymptotes intersect. F is the focus of the hyperbola and θ is the angle that OP makes with the vertical asymptote.

Figure 1 represents the theoretical learning curve of form (2). The accumulated errors are represented on $O'u'$, the trials on $O'R'$, O' being the point where training begins. P is the vertex of the curve. OR'_a is the horizontal asymptote of the curve and Ou'_a is the vertical

asymptote. O is the point at which the two asymptotes intersect. F is the focus of the hyperbola and the ratio $\frac{OF}{OP}$ is the eccentricity of the curve. ϑ is the angle that OP makes with the vertical asymptote.

From the first paper in this series the equations for the asymptotes of (2) are expressed by

$$S = u' = A + B \ , \tag{4}$$

$$R' = -C \ .$$

The coordinates of their point of intersection are $(-C, A+B)$. If the axes be translated to this point as the origin and if the coordinates of any point on the curve after translation be represented by (u'_a, R'_a) then

$$u' = u'_a + A + B \ ,$$

$$R' = R'_a - C$$

and substituting in (2) we have

$$u'_a + A + B = A + \frac{B(R'_a - C)}{C + R'_a - C} \ ,$$

or after simplification

$$u'_a R'_a + BC = 0 \ , \tag{5}$$

which is the equation of the *equilateral* hyperbola which lies in the second or fourth quadrant, when the axes of ordinates are the asymptotes of the curve and the origin lies at the intersection of the asymptotes.

The type form of the equilateral hyperbola is

$$2xy = c^2$$

where c is the length of the semi-major axis. The semi-major axis is represented by the distance OP in Figure 1. Therefore

$$c^2 = \overline{OP}^2 = -2BC \ .$$

But since (2) is an equilateral hyperbola we can use the conjugate or equivalent hyperbola in the 1st or 3rd quadrant and

$$\overline{OP}^2 = 2BC$$

or

$$\overline{OP} = \sqrt{2BC} \ . \tag{6}$$

Now in the case of the equilateral hyperbola the length of the semi-major axis, not the eccentricity of the curve, is the value of importance because of the fact that the eccentricity of all equilateral hyperbolas is $\sqrt{2}$. For any one curve the length of OP will always be the same no matter where the orginal lies. Therefore if we can find the value of OP in terms of the constants of (1) we can find the relationship between the constants of (1) and (2) and thus find the value of the learning constant K.

Examining (1) for its asymptotes we have

$$u = \frac{\sqrt{m}}{aK}\ ,\ R = \frac{-\sqrt{m}}{K}\ , \tag{7}$$

and

$$\left(\frac{-\sqrt{m}}{K}\ ,\ \frac{\sqrt{m}}{aK}\right)$$

will be the coordinates of the intersection of the asymptotes. If the axes be translated to this point as a new origin and if the coordinates of any point on the curve after translation be represented by (R_a, u_a) then

$$R = R_a - \frac{\sqrt{m}}{K}\ ,$$

$$u = u_a + \frac{\sqrt{m}}{aK}\ .$$

Substituting in (1)

$$u_a + \frac{\sqrt{m}}{aK} = \frac{\sqrt{m}}{aK} - \frac{\sqrt{m}}{K}\left\{\frac{u_a + \dfrac{\sqrt{m}}{aK}}{R_a - \dfrac{\sqrt{m}}{K}}\right\}$$

and after simplification

$$u_a R_a + \frac{m}{aK^2} = 0 \tag{8}$$

and

$$\overline{OP}^2 = \frac{2m}{aK^2} = 2BC\ .$$

Thus

$$\overline{OP} = \frac{1}{K}\sqrt{\frac{2m}{a}} \tag{9}$$

in which a is an arbitrary constant that can be included in the value of m. Then for Maze I we have

$$(OP)_1 = \frac{1}{K}\sqrt{2m_1}\ ,$$

for Maze V

$$(OP)_5 = \frac{1}{K}\sqrt{2m_5}\ .$$

Dividing

$$\frac{(OP)_1}{(OP)_5} = \sqrt{\frac{m_1}{m_5}}\ .$$

If we let the value of m_5 be our unit of measurement we have

$$(OP)_1 = \sqrt{m_1}\,(OP)_5\ , \qquad (10)$$

which is the equation of the best fitting straight line put through the values of $(OP)_1$ plotted against $(OP)_5$. The value $\sqrt{m_1}$ is the slope of this line and we have a solution for the difficulty of the maze.

Since we have incorporated a in the value of m we can solve for K from (9)

$$K = \frac{\sqrt{2m}}{OP}\ . \qquad (11)$$

Then for Maze I we have the value of K

$$K = \frac{\sqrt{2m_1}}{(OP)_1}\ , \qquad (12)$$

and for Maze V

$$K = \frac{\sqrt{2m_5}}{(OP)_5} = \frac{\sqrt{2}}{(OP)_5}\ . \qquad (13)$$

The two values of K should give a linear relationship.

We have in (10) and (11) solutions for the learning constant of the animal and for the relative difficulty of the two mazes independent of the initial adjustment period. These two values are affected only by the actual learning of the animals.

The vertex of the curve is a point of great importance due to the fact that at this point all of the animals are making errors at a rate of one error for each trial. We can establish this fact very simply.

The first derivative of (2) is

$$\frac{dw'}{dR'} = \frac{BC}{(C+R')^2}\ . \qquad (14)$$

If (R'_p, u'_p) represent the coordinates of the vertex,

$$\frac{R'_p + C}{OP} = \sin \vartheta \ ,$$

and

$$\frac{(A+B) - u'_p}{OP} = \cos \vartheta \ .$$

But ϑ is a 45 degree angle and

$$\sin 45 = \cos 45 = \frac{1}{\sqrt{2}} \ ,$$

and

$$R'_p = \frac{OP}{\sqrt{2}} - C \ ,$$

$$u'_p = (A+B) \frac{OP}{\sqrt{2}} \ . \tag{15}$$

Substituting these values in (14), and remembering that

$$BC = \frac{\overline{OP}^2}{2} \ ,$$

$$\frac{du'}{dR'} = \frac{\frac{\overline{OP}^2}{2}}{[C + (\frac{OP}{\sqrt{2}} - C)]^2} \ .$$

$$\frac{du'}{dR'} = 1.00.$$

Every animal is therefore making one error for each trial.

The vertex is a point that is comparable from one animal to the next. It is also comparable from one learning situation to the next. We can interpret the values of the coordinates at the vertex as the number of trials the animal must run before he makes one error per trial, and the number of errors it is necessary for him to run before he reaches the point at which he makes one error per trial.

It is possible to have at least two situations in which the animal never reaches the point where he makes one error per trial. First, he is so poor a learner that he fails to reach the vertex before training is discontinued; and second, he may learn the problem while he is still making more than one error per trial. The latter case may be accounted for in two ways; (a) the problem may be too easy to con-

TABLE 1
Critical Values of the Learning Curve—Maze I

No.	Lesion	A	B	C	S $(A+B)$	$\overline{OP}$ $(\sqrt{2BC})$	R'_p	u'_p	K
1	3.0	—6.06	47.25	7.46	41.2	26.6	11.3	22.4	0.040
2	3.0	—8.70	53.34	3.43	44.6	19.1	10.1	31.1	0.056
3	3.3	11.31	43.47	26.56	54.8	48.1	7.4	20.8	0.022
4	4.9	55.51	—15.82	—2.08	39.7	8.1	7.8	34.0	0.13
5	5.7	—8.55	65.95	8.32	57.4	33.1	15.1	34.0	0.033
6	9.8	137.48	—60.90	—3.21	76.6	19.8	17.2	62.6	0.054
7	9.9	—2.72	24.47	2.65	21.8	11.4	5.5	13.7	0.094
8	10.9	11.26	46.26	4.27	57.5	19.9	9.8	43.4	0.054
9	13.4	—47.00	106.60	3.60	59.6	27.7	16.0	40.0	0.039
10	14.1	—27.71	77.45	5.47	49.7	29.1	15.1	29.1	0.037
11	14.3	—19.78	54.72	2.25	34.9	15.7	8.9	23.8	0.069
12	15.2	—223.28	668.63	5.60	445.4	86.5	55.6	384.2	0.012
13	17.9	1.43	227.32	40.73	228.8	136.1	55.5	132.6	0.0079
14	18.4	—20.62	158.24	8.06	137.6	50.5	27.6	101.9	0.021
15	19.1	304.03	—190.99	—1.16	113.0	21.1	16.1	98.1	0.051
16	19.3	—3.46	42.91	2.91	39.5	15.8	8.3	28.3	0.068
17	19.6	—15.67	83.28	5.33	67.6	29.8	15.8	46.5	0.036
18	20.7	—122.52	370.14	13.80	247.6	101.1	57.7	176.1	0.011
19	21.0	12.42	85.09	10.69	97.5	42.7	19.5	67.3	0.025
20	21.9	9.40	245.54	23.93	254.9	108.4	52.8	178.2	0.0099
21	24.6	8.56	354.40	28.59	363.0	142.4	72.1	262.3	0.0076
22	25.2	—19.20	67.00	3.78	47.8	22.5	12.1	31.9	0.048
23	28.3	76.02	548.37	39.89	624.4	209.2	108.0	476.5	0.0051
24	31.4	—51.95	362.52	10.06	310.6	85.4	50.3	250.2	0.013
25	31.9	22.91	181.69	37.66	204.6	117.0	45.0	121.9	0.0092
26	34.7	34.99	1934.68	289.45	(1969.7)	(1058.3)	(458.9)	(1221.4)	(0.0010)
27	36.6	—871.45	1547.31	7.05	675.9	147.7	97.4	571.5	0.0073
28	36.8	—55.40	337.86	4.19	282.5	53.2	33.4	244.9	0.020
29	41.7	2523.76	—1416.80	—10.54	(1107.0)	(172.8)	(132.7)	(984.8)	(0.0062)
30	65.3	—177.14	1814.55	27.14	1637.4	313.8	194.8	1415.5	0.0034

Table 1. Column one is the number of the animal. Column two the percent of cerebral cortical injury. A, B, C are the constants of the learning curve. The horizontal asymptote, semi-major axis, trials at the vertex, errors at the vertex and the learning constant of each animal are represented by S, OP, R'_p, u'_p and K, respectively.

TABLE 2
Critical Values of the Learning Curve.—Maze V

No.	Lesion	A	B	C	S $(A+B)$	$\overline{OP}$ $(\sqrt{2BC})$	R'_p	u'_p	K
1	3.0	86.57	—23.78	—1.05	62.8	7.1	6.1	57.8	0.20
2	3.0	9.41	32.32	18.23	41.7	34.3	6.1	17.4	0.041
3	3.3	38.41	116.44	5.13	154.9	34.6	19.4	130.4	0.041
4	4.9	—92.59	235.14	2.33	142.6	33.1	21.1	119.2	0.043
5	5.7	6.00	86.27	5.09	92.3	29.6	15.8	71.4	0.048
6	9.8	42.97	192.93	5.43	235.9	45.8	27.0	203.5	0.031
7	9.9	43.09	56.57	10.65	99.7	34.7	13.9	75.2	0.041
8	10.9	227.39	—184.17	—0.56	43.2	14.4	10.8	33.0	0.098
9	13.4	6.09	153.94	3.61	160.0	33.3	19.9	136.5	0.042
10	14.1	33.06	155.44	38.67	188.5	109.6	38.8	111.0	0.013
11	14.3	—119.36	520.80	3.60	401.4	61.2	39.7	358.1	0.023
12	15.2	230.80	508.92	17.88	739.7	134.9	77.5	644.3	0.010
13	17.9	14.20	338.55	43.17	352.8	171.0	77.7	231.9	0.0083
14	18.4	74.24	132.94	32.36	207.2	92.8	33.2	141.6	0.015
15	19.1	32.98	90.49	36.18	123.5	80.9	21.0	66.3	0.017
16	19.3	119.38	—24.26	—10.88	95.1	23.0	27.2	78.8	0.061
17	19.6	42.38	38.16	6.87	80.5	22.9	9.3	64.3	0.062
18	20.7	34.29	750.45	24.95	784.7	193.5	111.9	647.9	0.0073
19	21.0	—0.68	141.58	33.33	140.9	97.1	35.4	72.2	0.015
20	21.9	105.03	334.58	9.36	439.6	79.1	46.5	383.7	0.018
21	24.6	—26.33	636.56	63.17	610.2	283.6	137.3	409.7	0.0050
22	25.2	—11.83	331.66	12.73	319.8	91.9	52.3	254.8	0.015
23	28.3	408.00	559.48	18.37	967.5	143.4	83.0	866.1	0.0099
24	31.4	—231.75	778.79	2.71	547.0	65.0	43.3	501.0	0.022
25	31.9	76.77	228.94	29.66	305.7	116.5	52.7	223.3	0.012
26	34.7	131.90	601.39	41.98	(733.3)	(224.7)	(116.9)	(574.4)	(0.0063)
27	36.6	47.30	1106.92	15.67	1154.2	186.3	116.3	1022.5	0.0076
28	36.8	—37.31	413.74	18.58	376.4	124.0	69.1	288.7	0.011
29	41.7	82.12	9936.88	602.81	(10019.0)	(3461.2)	(1844.5)	(7571.6)	(0.00041)
30	65.3	425.88	1063.14	15.43	1489.0	181.1	112.7	1360.9	0.0078

Table 2. Column one is the number of the animal. Column two the percent of cerebral cortical injury. A, B, C are the constants of the learning curve. The horizontal asymptote, semi-major axis, trials at the vertex, errors at the vertex and the learning constant of each animal are represented by S, OP, R'_p, u'_p and K, respectively.

stitute a good learning situation, for quantitative purposes, or (b) some form of sudden learning such as "insight" may cause a shift from many errors per trial to no errors per trial.

Tables 1 and 2 represent the various critical values of the learning curve. The first column gives the experimental number of the animal, the second represents the percent of cortical destruction in each case. A, B, and C are the constants of the learning curve (2) obtained in the former paper ('37). S represents the horizontal asymptote of the curve, OP is the length of the semi-major axis, R'_p is the number of trials at the vertex, u'_p is the number of errors at the vertex and K is the calculated learning constant of the animal.

S is calculated from equation (4)

$$S = u' = A + B \ .$$

OP is obtained from (6)

$$OP = \sqrt{2BC} \, ,$$

R'_p and u'_p from (15)

$$R'_p = \frac{OP}{\sqrt{2}} - C \ ,$$

$$u'_p = (A + B) - \frac{OP}{\sqrt{2}} \ .$$

In obtaining the value, K, we first plotted the values of $(OP)_1$ against $(OP)_5$. This relationship is represented in figure 2. We then solved for the slope of the best fitting straight line between $(OP)_1$ and (OP), by the method of least squares, obtaining the relationship

$$(OP)_1 = 0.7611 (OP)_5 \ .$$

From (10) we find that the value of $\sqrt{m_1}$ is 0.7611 and that the value of m_1 is 0.5793.

From the data for Maze I K was calculated from (12)

$$K = \frac{\sqrt{2m_1}}{(OP)_1} = \frac{1.0763}{(OP)_1} \, ,$$

and from the data for Maze V

$$K = \frac{\sqrt{2m_5}}{(OP)_5} = \frac{\sqrt{2}}{(OP)_5} \ .$$

The relationship between the values of K obtained from the two sets of calculations is shown in figure 5. The line represents the ideal

situation in which the values of K from the two mazes are equal. Our data group themselves around the line well enough to indicate linearity between the two values. The points which deviate most from the line represent the constants of those animals with very small lesions. Actual inspection of the original learning curves as they are published in the former paper will bear out this statement. As the lesions become more extensive, as the learning problem becomes more difficult, we find that the points tend to come nearer to the theoretical line. We will be able to put this to further test upon the completion of the fitting of the curves for our remaining data. The mazes for groups, II, III, and IV are more difficult than is the maze for group I. We, also, have normal animals trained on each maze.

Figure 3 represents the relationship between the trials at the vertex for Maze I and the trials at the vertex for Maze V. That there is a definite relationship is apparent from the figure. We are hoping to determine the nature of the relationship upon the completion of the calculations for the rest of our data.

Figure 4 shows the relationship between the accumulated errors at the vertex for Maze I and the accumulated errors at the vertex for Maze V. Here we have a very definite linear relationship. Remembering that the vertex represents that point at which all of the animals are making one error per trial, we can interpret this linearity as meaning that there is a high degree of relationship between the number of errors that the animal must make on Maze I, before he makes one error per trial, and the number of errors he must make on Maze V, before he makes one error per trial. In other words we have a point at which the animal makes errors at a constant rate and this point bears a constant relationship from one learning situation to another of the same type. The initial adjustment period of the animal is not eliminated from this measure.

Conclusions:

1. A solution for the difficulty of the learning problem and for the learning constant of the subject has been developed that is independent of the initial adjustment period of the subject.

2. The learning curve developed by L. L. Thurstone is an equilateral hyperbola and therefore has very definite attributes.

a. The semi-major axis represents the entire learning situation in one constant. It is a measure of the curvature of the learning curve. It is independent of the point from which learning is measured.

b. The vertex of the hyperbola represents the point at which the subjects are all making one error for each trial and is therefore

a point of comparison not only between subjects but between learning problems. The accumulated errors show a definite linear relationship from one maze to the other. The point at which the subject makes one error per trial on one problem is in constant relationship to the point at which he makes one error per trial on another problem of the same sort. The initial adjustment period is not eliminated in so far as the quantities are measured from the point at which training begins.

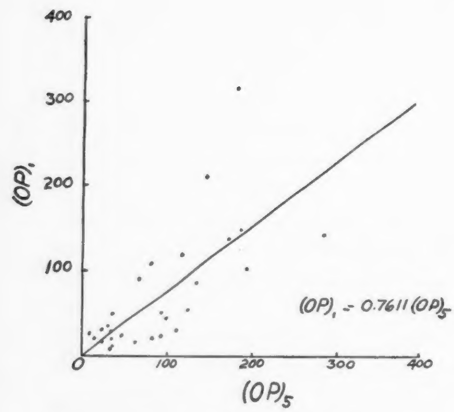

Figure 2. The relationship between the semi-major axis on Maze I, $(OP)_1$, and the semi-major axis on Maze V, $(OP)_5$.

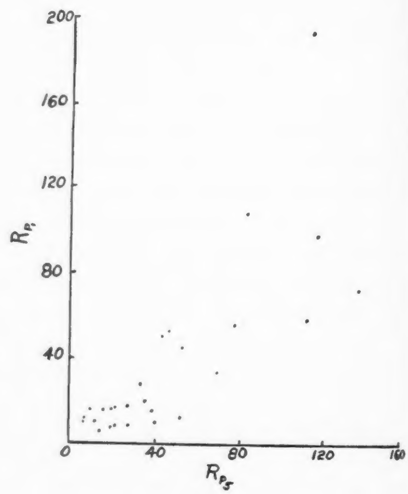

Figure 3. The relationship between the trials at the vertex on Maze I, R_{p_1}, and the trials at the vertex on Maze V, R_{p_5}.

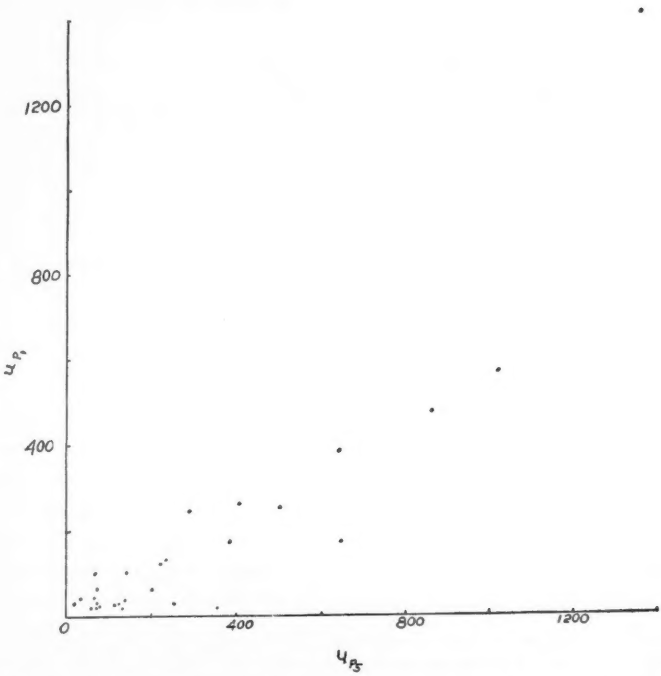

Figure 4. The relationship between the accumulated errors on Maze I, u_{p_1}, and the accumulated errors on Maze V, u_{p_5}.

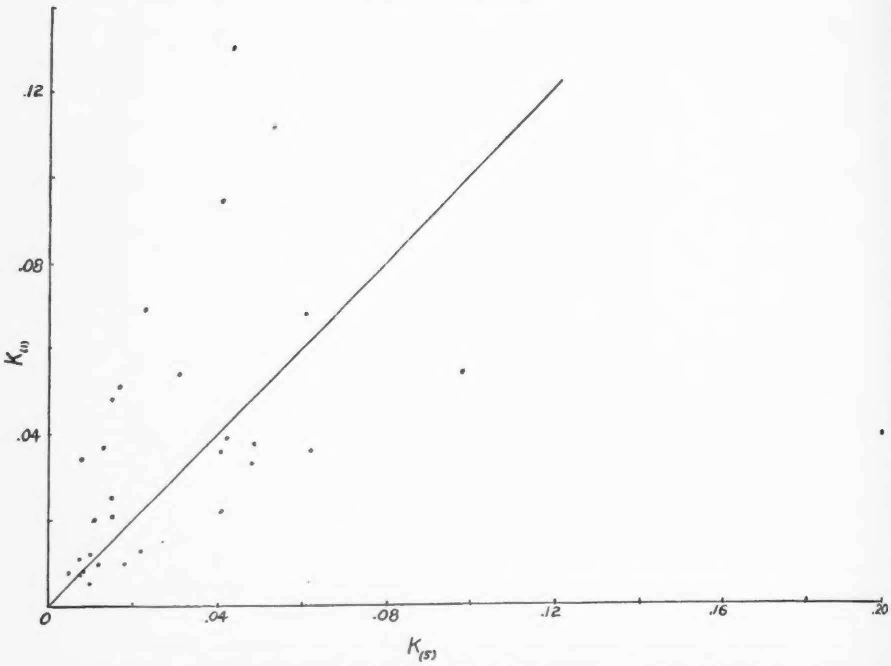

Figure 5. The relationship between the learning constant calculated from the data taken on Maze I and the learning constant calculated from the data taken on Maze V.

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THE PSYCHOPHYSICS OF MENTAL TEST DIFFICULTY

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By the use of the Seashore tests of Pitch Discrimination, Intensity Discrimination, Time Discrimination, and the test of Tonal Memory, it is shown that the easiness of an item, as determined by absolute scaling methods, is proportional to the logarithm of the magnitude of the stimulus. It is proposed that this is a case of Fechner's psychophysical law and that the unit of absolute scaling as applied to test items may become a satisfactory unit of all *S*-scales in the more traditional psychophysical problem.

In very recent years we have seen more and more clearly the common ground, rationally and statistically, that exists between psychophysical and mental-test problems. This common territory would have been more readily envisaged, perhaps, had we been able to evaluate test items on any type of physical scale. In most cases we are unable to this, and as a matter of fact the question of the physical difficulty of a test item has had little or no practical importance. But the theoretical importance is not to be overlooked. In a very real sense any psychophysical judgment may be regarded as a response to a test item if only we are interested in whether the organism that renders it is correct in his response, *i.e.*, whether his judgment coincides with the "real" facts of the stimulus situation.

For some time we have been able to evaluate test items on a scale of psychological difficulty. The psychophysical question is whether or not there is any consistent or universal functional relationship between the psychological difficulty of an item and some corresponding physical evaluation of it. It is the purpose of the present paper to inquire into this possibility. The tests that lend themselves to this investigation most readily are of the simple psychophysical variety. The Seashore Tests of Musical Talent include material of this sort. The gradations of difficulty are determined by variations in stimulus differences that are definitely measurable on some physical scale, *viz.*, frequency, in double vibrations per second, intensity of sounds in audiometer units, and time intervals in fractions of a second. We may also include in this consideration the test of tonal memory, correlating the difficulty of the test item with the number of tones in the melody,

although the measurements of physical gradation in this test are not so satisfactory as they might be.

The data that are to furnish the material of this study, then, are the results from these four Seashore tests, including judgments of (1) pitch, (2) intensity, (3) time, and (4) judgments of altered tones in the test of tonal memory. These tests have been given year after year in the University of Nebraska elementary laboratory course under the standard test conditions prescribed for them. The subjects were undergraduates with the usual range of talent exhibited in the tests by that type of subject. The numbers of subjects taking the four tests vary, since in some years one or more of the tests were omitted and always a few papers have to be rejected because the subjects failed to follow instructions or did not complete the test. The numbers of papers used in each test were as follows: Pitch, 570; Intensity, 439; Time, 123; and Memory, 412.* As a check upon the results obtained from these data, some comparable data published by Farnsworth were employed.† These data were obtained from two groups of 100 each at Stanford University.

In determining item difficulty, all items of equal physical magnitude in a test were treated together. Every test is divided into sets of ten stimuli of homogeneous magnitude. For every set of ten the proportion of correct judgments was determined. From this proportion was found the most probable proportion of successes when a correction for chance successes is made. The formula, which was published previously by the writer, reads as follows:‡

$$_cp = \left(\frac{n}{n-1}\right) p - \left(\frac{1}{n-1}\right) ,$$

in which $_cp$ = the proportion of successes corrected for chance,

p = the uncorrected proportion,

and n = the number of alternative responses to the item.

Since in all except the test for tonal memory $n = 2$, the correction for three of the tests reduced to $_cp = 2p - 1$.

*For the use of these data I am indebted to Miss Kathleen Carter who gives the proportions of correct judgments for every level of difficulty of item in a Master's thesis entitled "A weighted scoring scale for the Seashore tests of musical ability," 1934, on file in the University of Nebraska Library.

†P. R. Farnsworth. "A critical study of the Seashore-Kwalwasser test battery," *Genet. Psychol. Monog.*, 1931, **9**, 291-393.

‡J. P. Guilford. "The determination of item difficulty when chance success is a factor," *Psychometrika*, 1936, **1**, 259-264.

From this point, the determination of the difficulty (or ease) of passing an item of a given type was as usual. The proportion was assumed to represent the surface under the normal distribution curve below a certain deviate on the abscissa. The deviate represents the position of the item on the psychological scale and it is denoted by the customary symbol S. For the purposes of this study the scale value of the item is given in terms of ease of passing. Easy items will then have higher (more positive) values, and difficult items will have lower (more negative) values. Table 1 gives the original uncorrected

TABLE 1

Proportions of Successes and the Corresponding Scale Values for Items in the Four Seashore Tests of Musical Talent.

PITCH DISCRIMINATION

	Carter's Data		Farnsworth's Data						Best Fitting S
R	p	S	p	S	S'	p	S	S'	S''
.5	.5165	—1.838	.488*	(—2.000)	(—1.792)	.471*	(—2.000)	(—1.798)	—1.600
1	.5668	—1.108	.550	—1.282	—1.194	.548	—1.305	—1.204	—1.096
2	.6218	— .604	.652	— .513	— .553	.648	— .536	— .546	— .592
3	.7016	— .246	.731	— .095	— .205	.733	— .085	— .161	— .297
5	.8192	.353	.864	.608	.381	.866	.619	.441	.075
8	.8596	.580	.911	.923	.643	.895	.806	.601	.417
12	.8856	.742	.941	1.185	.861	.914	.946	.721	.712
17	.9146	.950	.938	1.155	.836	.914	.946	.721	.965
23	.9412	1.185	.964	1.461	1.091	.961	1.412	1.119	1.185
30	9447	1.221	.963	1.447	1.079	.971	1.572	1.256	1.379

INTENSITY DISCRIMINATION

	Carter's Data		Farnsworth's Data						Best Fitting S
R	p	S	p	S	S'	p	S	S'	S''
1	.7546	.023	.746	— .020	.065	.724	— .133	.018	— .031
2	.8368	.450	.872	.656	.451	.882	.719	.495	.495
3	.9151	.955	.929	1.728	1.063	.937	1.943	1.181	.802
4	.9480	1.259	.994	2.257	1.365	.990	2.054	1.243	1.020
5	.9452	1.229	.988	1.977	1.205	.978	1.706	1.048	1.190
1	.7649	.075	.721	— .416	— .161	.716	— .171	— .003	— .031
2	.7868	.185	.831	.418	.315	.802	.264	.241	.495
3	.8960	.813	.922	1.011	.653	.941	1.185	.756	.802
4	.9010	.849	.968	1.522	.945	.960	1.405	.880	1.020
5	.9345	1.122	.976	1.706	1.050	.981	1.774	1.086	1.190

*Proportions of .500, or ones slightly less like the two in this table, lead to scale values of minus infinity, a scale value that is theoretically reserved for stimulus differences of zero. It has been assumed that the proportions of .488 and .471 are due to sampling errors. In groups of only 100 subjects a reasonable lower limit for S here is about —2.000, and that assumption has been made.

(TABLE 1, continued)
TIME DISCRIMINATION

	Carter's Data		Farnsworth's Data						Best Fitting S
R	p	S	p	S	S'	p	S	S'	S''
.02	.5547	—1.232	.561	—1.165	—1.183	.570	—1.080	—1.056	—1.189
.02	.5749	—1.036	.595	— .878	— .923	.565	—1.126	—1.099	—1.189
.05	.6584	— .476	.704	— .233	— .339	.655	— .496	— .510	— .294
.05	.7025	— .240	.678	— .369	— .462	.653	— .507	— .520	— .294
.09	.7734	.118	.816	.337	.178	.826	.391	.320	.280
.09	.7857	.179	.785	.176	.032	.789	.197	.139	.280
.14	.8768	.687	.894	.800	.597	.900	.842	.742	.711
.14	.9027	.860	.899	.834	.628	.905	.878	.776	.711
.20	.9150	.954	.963	1.447	1.183	.950	1.282	1.153	1.059
.20	.9532	1.316	.978	1.706	1.418	.953	1.316	1.185	1.059

TONAL MEMORY

	Carter's Data		Farnsworth's Data						Best Fitting S
R	p	S	p	S	S'	p	S	S'	S''
2	.945	1.226	.964	1.461	1.256	.956	1.353	1.277	1.246
3	.878	.904	.913	1.126	.954	.865	.834	.842	.941
4	.822	.716	.823	.719	.587	.794	.598	.635	.580
5	.624	.075	.662	.179	.116	.618	.055	.159	.138
6	.439	— .448	.448	— .418	— .439	.388	— .656	— .464	— .431

proportions of successes for all the data with the scale values derived from them.

In the case of the first three tests we should expect to find a positive correlation between a test item and its corresponding physical magnitude, since the greater the physical difference, the easier the task. In the case of the test of tonal memory we should expect a negative correlation, since the greater the number of tones in the melody the less easy the task. Both of these expectations are fulfilled, but the regressions are not rectilinear in any case.

It will be noticed in Table 1 that, in general, Farnsworth's subjects gave a wider range of scale values for the test items than did Carter's subjects. According to the principles of absolute scaling, this means that the absolute variability of his groups was less than that in Carter's groups.* This is partly due to the fact that Carter's groups were numerically larger, but apart from that fact there may have been a genuine difference in scatter of talent. If we adopt the standard deviation of the distribution of ability in Carter's groups as the

*For a brief description of the absolute scaling processes and their underlying principles, see J. P. Guilford, *Psychometric Methods.* New York: McGraw-Hill, 1936, 440-443.

absolute unit of the scale, then the corresponding variabilities of Farnsworth's groups are .883 and .885 in the test of pitch discrimination, .571 and .560 in intensity discrimination, .906 and .936 in time discrimination, and .902 and .877 in tonal memory. In every case, the pair of values is gratifying uniform, showing that the two groups of 100 subjects each were markedly equivalent so far as variability is concerned. In general, the means of the scale values for Carter's groups were lower than those in the other groups. Table 2 summarizes these differences, giving the means and standard deviations of the various groups in terms of the absolute scale adopted. By means of the usual equations employed in scaling,† the *S*-values contributed by Farnsworth's data were recast in terms of the absolute scale. These new values are listed under *S′* in Table 1.

TABLE 2

Means and Variabilities of the Different Groups of Subjects in the Four Seashore Tests, Given in Terms of Absolute Scaling.

Test	Carter's Groups		Farnsworth's Groups			
	Mean	σ	Mean	σ	Mean	σ
Pitch	.000	1.000	.126	.883	.088	.885
Intensity	.000	1.000	—.076	.571	—.093	.560
Time	.000	1.000	.128	.906	.046	.935
Memory	.000	1.000	.062	.902	—.111	.877

†*Ibid*, 442.

Figures 1-3 show the scale values of the items plotted against the stimulus values for the first three tests. The Carter values are designated by circlets and the Farnsworth values by crosses. The next obvious question concerns the nature of the functional relationship existing between *S* and *R*. From the standpoint of traditional psychophysics, the Fechnerian *S*—*R* relationship immediately suggests itself. From an inspection of the trend in Figs. 1-3 this suggestion receives strong encouragement. The fact that a test item having a physical difference of zero should be infinitely difficult also lends support to the idea that Fechner's law may reasonably apply here. Accordingly, I have fitted logarithmic functions to the data in the first three tests, using the method of least squares, without weighting the observations, for the Carter data alone and for all the data combined. The parameters of the equations and the indices of correlation are summarized in Table 3 and the lines of best fit are shown in Figs. 1-3. From the equations involving these parameters the theoretical scale values, *S″*, were computed and they appear in the last column of Table 1.

The test of tonal memory is not so easy to deal with in the manner of the first three. Let us assume that the difficulty of the melody is measurable physically in terms of the number of tones it contains.

TABLE 3

Parameters and Indices of Correlation Between the Scale Values of the Test Items and the Magnitudes of the Stimuli in the Seashore Tests.

	a	*b*	*p*	*a*	*b*	*p*
Pitch	—1.136	1.734	.990	—1.096	1.675	.957
Intensity	— .077	1.727	.942	— .031	1.746	.952
Time	2.672	2.285	.942	2.613	2.248	.971
Memory	—2.640	4.608	.994	—2.608	4.560	.990

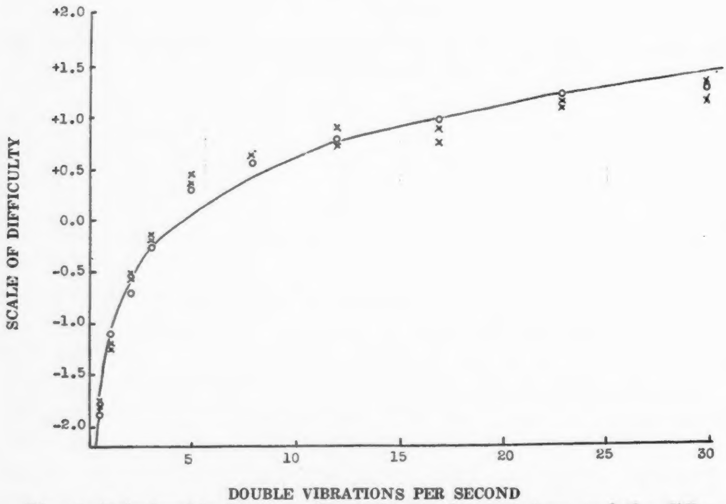

Fig. 1. Relationship between the easiness of a test item and the difference in frequency in the Seashore test of pitch Discrimination.

There is no other reasonable or practical unit in this case.* In keeping with the treatment of the other tests, let the zero point be at a place where the probability of passing the item is infinitely small. The practical question is, how long must a melody be before the subjects absolutely fail? By absolute failure we can only mean that the pro-

*It is to be questioned whether all ten melodies of equal length are really equal in difficulty since the melodies themselves differ in slope, in form, and in the position of the altered tone. But we may assume that the average difficulty of a set of ten of the same length is a representative level for that number of tones.

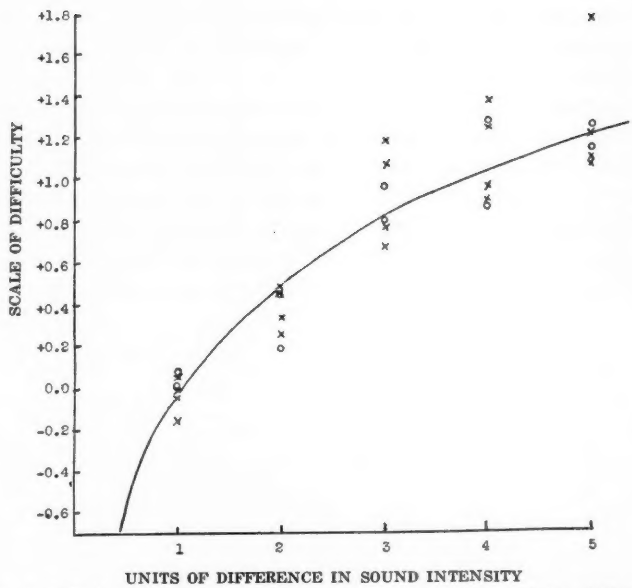

Fig. 2. Relationship between the easiness of an item and the difference in intensity of two sounds in the Seashore test of intensity discrimination.

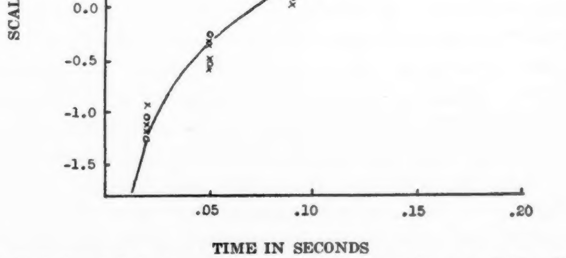

Fig. 3. Relationship between the easiness of an item and the difference between two time intervals in the Seashore test of time discrimination.

portion of correct responses is no greater than could most likely occur by pure guessing. With a limited number of tones, even with 10 to 12, the proportion of successes to be expected by chance is noticeably greater than zero.

For a rough solution to the problem, let us examine the two curves in Fig. 4. The upper curve is drawn through the proportions of successes actually obtained in the three groups of subjects. The lower curve is drawn through the probabilities of chance successes at the different lengths of melody. The point of intersection of these two curves may be taken tentatively as the point of real zero success and

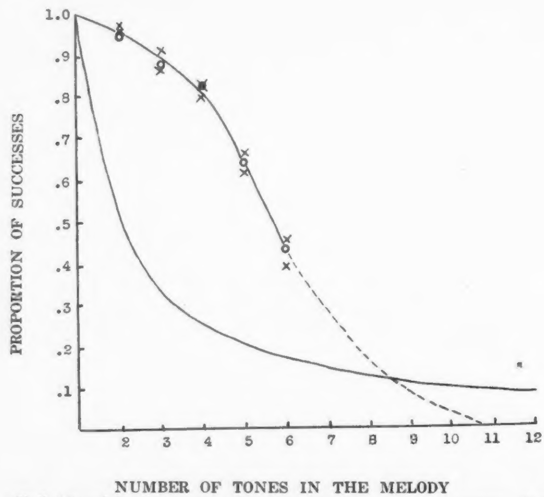

Fig. 4. Relationship between the proportions of successes, both actual and expected by pure chance, and the length of melody in the Seashore test of tonal memory.

so of infinite difficulty. Obviously, the relatively small values of the Seashore melodies do not reach this point. At least 40 per cent of the judgments are correct for the longest melody employed, the one with 6 tones. Even a tentative solution here requires an extrapolation. As a guess, it is assumed that the upper curve in Fig. 4 is actually symmetrical and the dotted continuation has been drawn on that assumption. It is quite possible that the upper line becomes asymptotic to the lower one as the two are continued to the right. In this case, however, one would have to select an arbitrary limit for practical purposes. Under the assumption made above, a melody of 9 tones was adopted as the zero point of the physical scale. The logarithmic relationship

was assumed to apply and an equation of the form $S = a + b \log(9 - R)$ was derived by the method of least squares. The parameters of the equation appear in Table 3, and the index of correlations (.994 for the Carter data and .990 for all the data) seem to bear out the choice of type of function and of the zero point. Other choices of zero point might prove just as satisfactory, but the choice of 7 and of 13 led to lower correlations than that obtained with the choice of 9. It might be thought that a function of the type $S = a + b \log (1/Y)$ would be satisfactory. This possibility has been examined. The plot of S against $\log(1/Y)$ shows a nonlinear regression, sufficient to reject this function. In any type of function for the memory test the relationship

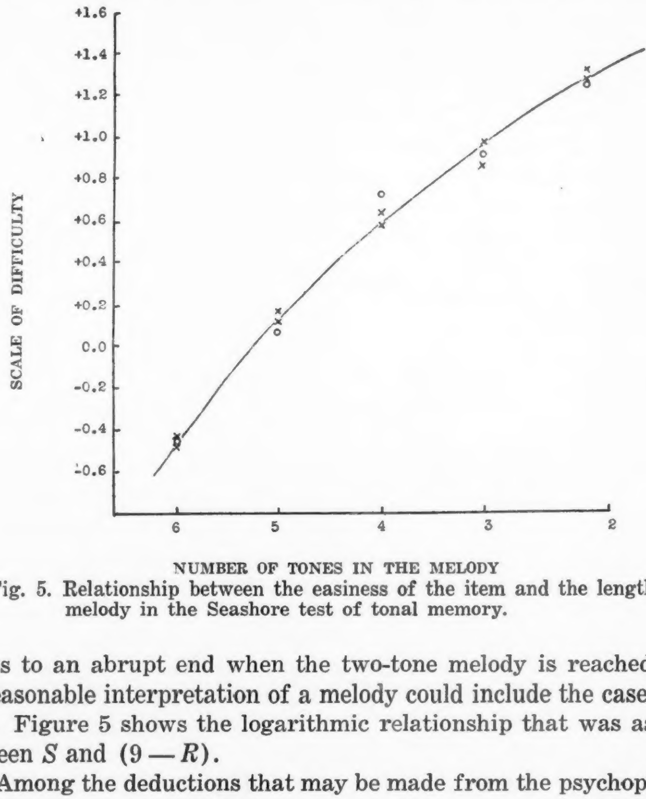

Fig. 5. Relationship between the easiness of the item and the length of melody in the Seashore test of tonal memory.

comes to an abrupt end when the two-tone melody is reached, since no reasonable interpretation of a melody could include the case of one tone. Figure 5 shows the logarithmic relationship that was assumed between S and $(9 - R)$.

Among the deductions that may be made from the psychophysical law that seems to exist between the easiness of a test item and its corresponding physical magnitude are several interesting and perhaps fruitful implications. In the first place, it would seem that in

any one of these tests (the test of tonal memory excluded) no difference can be large enough to become infinitely easy to pass. Given a sufficient number of trials, at least one subject may be expected to judge any difference incorrectly at least once no matter how large that difference is. This is not unreasonable or impossible. During a sudden and complete lapse of attention at a critical moment any difference, no matter how large, may be judged incorrectly. There is nothing in this hypothetical case that is not also found in any normal distribution of measurements. In the latter case, infinitely large deviations from the mean may occur, infinitely seldom, to be sure, but nevertheless they should by hypothesis occur.

Whether or not the logarithmic function is the correct one for the purpose, there is a law of diminishing returns. Equal increments of stimulus magnitude give diminishing increments of easiness of the item. This might be taken to mean that large differences in these tests are judged in a different manner than are small ones, different criteria or abilities being involved as the physical magnitude changes. Since in these tests stimuli of homogeneous magnitude are given in sets of ten, subjects could easily take on different attitudes during each ten. A check on this point should be made in which the different stimulus magnitudes are given in haphazard order. If the suggestion that different magnitudes of stimuli draw upon different primary abilities is taken seriously, then a factor analysis is in order. The writer is of the opinion that, in general, tests of homogeneous content and operations will not depend upon the same primary factors or abilities when they are difficult as when they are easy for the same population.

The logarithmic relationship here proposed may be purely and simply a case of Fechner's psychophysical law; that the S-scale, conceived as a continuum of test-item difficulty, is of the same species as the S-scale conceived as a continuum of sensory gradations. I am inclined toward this interpretation of the situation. It is yet to be shown whether or not the units of test-item scales are comparable with the units of scales obtained by paired comparisons, equal appearing intervals, or with the traditional j.n.d. which is derived from a number of the psychophysical methods. It is possible that the unit of the test-item scale is more nearly uniform than is the traditional j.n.d. which is now known not to be a psychological constant. Much of the data obtained from the traditional psychophysical methods can be treated according to the scaling methods used with test items. The method of absolute scaling therefore becomes an additional psychophysical method.

On the whole, it may not be premature to urge that the unit of increment of difficulty of any type of task should become the universal psychological unit. In this, at least, there is common meeting ground for very diverse types of psychological measurement. There are some obstacles in the way of a ready application of this principle, to be sure; the selection of the standard population and the variation of mental abilities depended upon for tasks of different variety and of different levels of difficulty. Aids to the solution of these problems will be found in the methods of absolute scaling and of factor analysis respectively.

As an illustration of the use of the test-item scaling device as a psychophysical method, let us consider the problem of finding differential limens from the Seashore test data. Such limens could be estimated by the method of right and wrong cases, but they can also be estimated from the logarithmic functions of best fit and their parameters as listed in Table 3. The *DL* in these tests may be defined as that stimulus difference of median difficulty. This definition of the threshold can apply whether we have applied the stimuli to a large number of subjects a limited number of times or to one subject a large number of times. The statistical question is to find the R-value when $S = 0$. Since $S = a + b(\log R)$, when $S = 0$, then

$$0 = a + b(\log R). \tag{1}$$

Transposing,

$$b(\log R) = -a$$

and

$$\log R = -\frac{a}{b}.$$

$$R_0 = \text{antilog}\ (-\frac{a}{b}) \tag{2}$$

where R_0 is the value of R when $S = 0$.

The *DL*'s for Carter's data are listed in Table 4. These include a limen for the test of tonal memory. This limen is hardly to be regarded as a *DL*; it is more in the nature of an *RL*. The value, 5.26, means that a hypothetical melody with 5.26 tones is of median difficulty for Carter's group. In order to compute this limen a slight modification had to be introduced in equation (2) so that it becomes $R_0 = 9 - \text{antilog}\ (-\frac{a}{b})$.

The thresholds for the Farnsworth data could be found of course by fitting logarithmic functions to his eight sets of observations. But

TABLE 4

Thresholds Computed from the Logarithmic Equations between S and R in the Seashore Tests.

Test	Carter's Data	Farnsworth's Data	
Pitch	4.52	3.74	4.00
Intensity	1.11	1.15	1.17
Time	.068	.060	.066
Memory	5.26	5.36	5.07

they may also be estimated from the equations already at hand. The S-values for his data were scaled in terms of the median and sigma of Carter's group. Equations of the logarithmic functions for all data combined were then determined. Let us assume that these equations apply to Farnsworth's two groups of subjects. But it will be seen in Table 2 that the median values of difficulty for those two groups did not coincide with the medians of the absolute scales. Let d stand for the deviation of a Farnsworth median from the absolute zero. We may then modify equation (1) to read

$$0 = a + b(\log R) + d \ .$$

$$b(\log R) = -a - d \ ,$$

$$\log R = -\frac{a+d}{b} \ ,$$

and

$$R_0 = \text{antilog}\left(-\frac{a+d}{b}\right) \ . \tag{3}$$

The limens for Farnsworth's data have been computed on this basis and they appear in Table 4.

The use of (log R) instead of R on the physical scale is a practice that should come more and more into use. Stimulus scales of different varieties although measurable in physical units should very frequently be translated into terms of logarithms before the computation of results is begun. The use of the geometric mean should often replace the use of the arithmetic mean. Frequency distributions that are positively skewed will often become symmetrical if equal logarithmic units are employed on the base line. This is in line with Thurstone's proposal of a phi-log-gamma function in the method of constant stimuli to replace the more traditional phi-gamma function.*

Incidentally, if it can be shown that the logarithmic relation be-

*L. L. Thurstone, The phi-gamma hypothesis, *J. Exper. Psychol*, 1928, **11**, 293-305.

tween S and R in the scaling of test items holds quite generally, then by means of this relationship we will be able to estimate the relative physical magnitudes of test items even when such "physical" magnitudes are difficult or well-nigh impossible to evaluate directly. We should be able, for example, to say that a certain item is as difficult as is the task of discriminating a certain difference in pitch or a certain difference in time. The comparability of stimuli and of stimulus differences from diverse fields would thus be assured. This would be of consequence, for example, in selecting stimuli from different sense departments to be used in comparative studies of adaptation time, reaction time, and similar problems. Other practical and theoretical implications of the psychophysics of test difficulty as set forth in this report will no doubt arise as time goes on. Further empirical work and attempts at rationalizing should bring us still nearer to the ideal of a single rational and operational basis of psychometric methods in general.

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NOMOGRAPH FOR POINT BISERIAL r, BISERIAL r, AND FOURFOLD CORRELATIONS

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A nomograph for the computation of point biserial r, biserial r, and fourfold correlation coefficients is presented, with instructions for use.

I. Use of Nomograph for Computation of Point Biserial r and Biserial r

Formulas and notation involved

$$r_{p\text{-}bis} = \frac{T-p}{\sqrt{pq}} \cdot \frac{M}{\sigma} = \frac{d}{\sqrt{pq}} \cdot K$$

$$r_{bis} = \frac{T-p}{z} \cdot \frac{M}{\sigma} = \frac{d}{z} \cdot K$$

$r_{p\text{-}bis}$ = point biserial coefficient of correlation.

r_{bis} = biserial coefficient of correlation.

$$T = \frac{\Sigma X_i}{\Sigma X_t}$$

ΣX_i = sum of scores of those who have passed the item.

ΣX_t = sum of scores of entire population.

p = proportion of population passing the item.

$d = T - p$

z = ordinate of normal curve corresponding to p

$$K = \frac{M}{\sigma}$$

M = mean of total distribution of scores.

σ = standard deviation of total distribution of scores.

1. *Point biserial r*

Data needed: d, p, and K.

Directions: Find the d and K values on the d and K scales of the nomograph. Place celluloid strip with straight line etched on it and with pin hole through center (such as that provided for Dunlap and Kurtz's *Handbook*) to connect the two values so that pin hole cuts the center line. Using a pin through the hole, pivot the strip so that line cuts the $p_{p\text{-}bis}$ scale at p. Point biserial r will be found where the line on the strip crosses the r scale. Note: the p scale does not go above .50. In case of larger proportions, subtract from 1.00 and use figure so obtained.

Example: $d = .034$, $p = .10$, $K = 2.2$. Place the line on the celluloid strip on the points corresponding to .034 on the d scale and 2.2 on the K scale. Pivoting the scale on the center line, move the left end until the line on the strip cuts the $p_{p\text{-}bis}$ scale at .10 . The resulting point biserial coefficient of correlation is read on the r scale as .25.

2. *Biserial r*

Data needed: d, p, and K

Directions: Directions for obtaining point biserial r are to be used with the exception that the value for p is located on the p_{bis} scale. The r so obtained is biserial r.

Example: Follow the directions for the example given for point biserial r, but locate p on the p_{bis} scale. The resulting value of r_{bis} is .43 .

II. Computation of Fourfold Correlations

Formulas and notation involved

$$\varphi = \frac{T - p}{\sqrt{pq}} \sqrt{\frac{p'}{q'}} = \frac{d}{\sqrt{pq}} \sqrt{\frac{p'}{q'}}$$

$$\Psi = \frac{T - p}{z} \sqrt{\frac{p'}{q'}} = \frac{d}{z} \sqrt{\frac{p'}{q'}}$$

φ = correlation in a fourfold point surface.

ψ = correlation of a point-distributed dichotomy with dichotomy formed by arbitrary division of a normally-distributed variable.

p = proportion of population passing first item.

p' = proportion of population passing second item.

$T = \frac{a}{p'}$ in which $a =$ proportion of population passing both items.

$d = T - p$

1. *Correlation in a fourfold point surface*

$z =$ ordinate of normal curve corresponding to p.

Data needed: d, p. and p'.

Directions: Find the d and p' values on the d and p' scales of the nomograph. Place the celluloid strip to connect the two values so that the pin hole cuts the center line. Using a pin through the hole, pivot the strip so that the line cuts the $p_{p\text{-}bis}$ scale at p. The value of φ will be found where the line on the strip crosses the r scale.

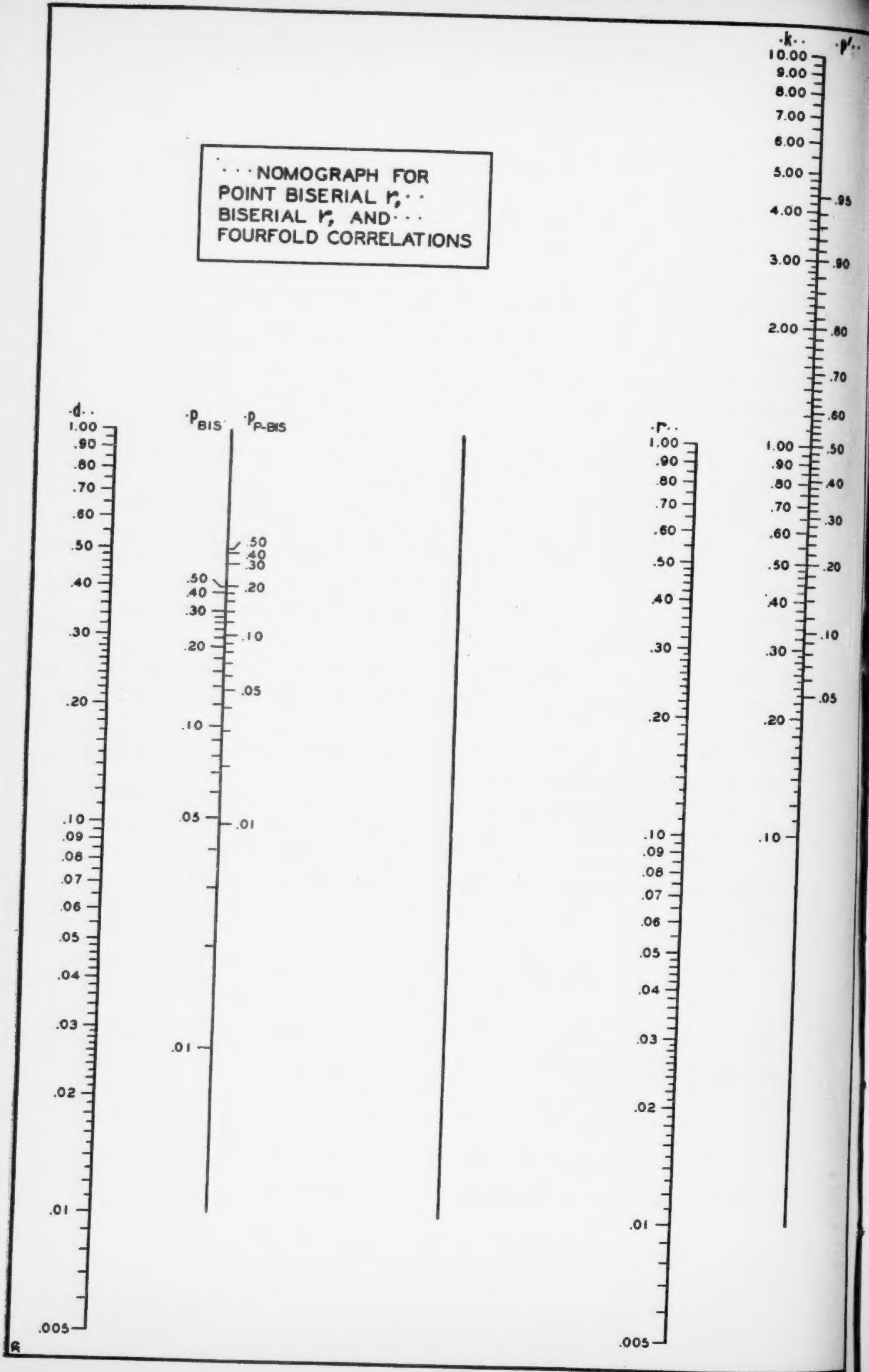
Example: $d = .23$, $p = .51$, $p' = .30$. Place the line on the celluloid strip on the points corresponding to .23 on the d scale and .30 on the p' scale. Pivoting the scale on the center line, move the left end until the line on the strip cuts the $p_{p\text{-}bis}$ scale at .49 (since p is more than .50, the value $1.00 - p$ is used). The resulting φ is read on the r scale as .30 .

2. *Correlation of a point-distributed dichotomy with dichotomy formed by the arbitrary division of a normally-distributed variable.*

Data needed: d, p, and p'. In this case, p must be the proportion for the normally-distributed variable.

Directions: Directions for obtaining φ are to be followed with the exception that the value for p is located on the p_{bis} scale.

Example: Follow the directions for the example given for φ, but locate p on the p_{bis} scale. The resulting value of ψ is .38.


NOMOGRAPH FOR
POINT BISERIAL r,
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d
p BIS
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r
k
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.95

.90

.80

.70

.60

.50

.40

.30

20

10

5